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GRAPHICAL METHODS
IN APPLIED MATHEMATICS

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PREFACE.

THE importance of Graphics in modern mathematical training, and its numerous uses in practical work, render unnecessary any excuse for the publication of an elementary account of some of its applications, provided these applications are chosen with discretion and treated with clearness.

The author is hopeful that competent judges will consider that the present book fulfils these requirements. It has not been written with a view to any particular examination; but the easier parts will be found to meet the needs of secondary schools and of candidates in military and naval examinations; while students in technical colleges and candidates in the examinations of the University of London will, it is believed, find most of the chapters of definite use to them.

All sections and exercises marked with an asterisk should be omitted in a first reading of the volume; students who wish further to curtail the course of work will find an easy First Course mapped out on page ix.

Special attention is directed to the large number of concrete examples, worked out in detail, which are supplied in the various chapters. It is essential that the student should himself work out the graphical constructions according to the instructions given, and afterwards compare his results with those obtainable by measurement of the figures in the text. To avoid the tendency to produce very small figures, which characterise the work of almost all students, the instructions supplied will be found to determine large drawings in nearly all cases. An endeavour should be made so to construct the diagrams that all lengths are correct to at least three numerical figures; it is hoped that this degree of accuracy has been attained in the answers given at the end of the book. Owing to slight, perhaps very slight, errors in construction the final result, obtained by measurement, will often be slightly incorrect in the third figure.

The student is strongly urged not to confine himself to graphical methods only in statics and mensuration. The employment of calculation and graphics may be likened to the use of our two hands; no matter how highly developed one instrument may be, much more can be done with the two conjointly than with one alone. Of necessity, in this book analytical methods and calculations are only incidentally touched upon, but students with a knowledge of Trigonometry will see that even roughly drawn vector polygons can easily be used for purposes of calculation.

This opportunity is gladly taken to acknowledge a debt of gratitude to Prof. Henrici, F.R S., of the Central Technical College, London to whom the author's first knowledge of the true value of Graphics is due. His teaching showed Statics and Dynamics not merely as a branch of somewhat unsatisfying Mathematics, but as a real and interesting subject with important applications. Those acquainted with Prof. Henrici's work and lectures will appreciate the author's obligation to him.

Thanks are also due to Mr. E. F. Witchell of the Central Technical and Goldsmiths' Colleges for reading most of the proof sheets, suggesting improvements, and correcting some of the answers; to Prof. R. A. Gregory and Mr. A. T. Simmons for their unsparing trouble during the preparation of the MSS. and while the book was passing through the press; and finally to the Senate of the University of London and the Controller of H.M. Stationery Office for permission to make use of problems set in various University, Civil Service, Naval, Military, and Board of Education examinations. The source of each such problem, and the date when subsequent to 1902, has been given after the question.

G. C. TURNER.

SIDCUP, *November*, 1907.

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INSTRUCTIONS TO THE STUDENT.

For the construction of the figures required in this book a set square with a 3-inch side is useless. The side of the 45° set square should be at least 6 or 7 inches.

The standard scale used should be flat on one side and bevelled on the other and the scale divisions should reach to the edge. One edge should be divided into fiftieths of an inch and the other into millimetres or half-millimetres. Scales of this description can be obtained from Messrs. Aston & Mander, Old Compton Street, London, W., and other makers, at 1s. 6d. each.

An angle is best set off or measured by means of its tangent or by a scale of chords. If a protractor is used it should be a large semicircular one of transparent material.

Hard chisel-pointed pencils should be used for all the constructions.

A FIRST COURSE OF STUDY.

Ch. I., pp. 1-17, 28-30, 33-35, 37-40.
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CHAPTER I.

GRAPHICAL ARITHMETIC.

Scalar Quantities. In Mechanics and Physics, quantities such as numbers, volumes, masses, time, temperature, displacement, velocity and force are dealt with. Some of these quantities are related to direction in space and cannot be defined without reference to direction, others have no such relation to space.

Mass, time, temperature, volume and number are examples of quantities which are completely given when we know the kind of quantity and how much there is of it; they are called **Scalar Quantities.**

To specify the amount needs reference to some unit, a gramme, a degree centigrade, a cubic centimetre, the number 1 ..., so that Scalar Quantities are specified by giving

(1) the unit quantity, (2) the number of units.

Vector Quantities. Those quantities which require for their specification some reference to direction in space are called **Vector Quantities.** Examples of these are displacement, velocity, acceleration, force, ...

An hour differs from a minute only in amount, but the pull of the earth on a book differs from the pull of a locomotive on a train not only in amount but also in direction.

Time is a scalar quantity and force a vector quantity.

Quantities to Scale. The word **scalar** is used because these quantities can be graphically represented to scale by lengths (Latin *scalae*—a ladder—divided into equal parts by the rungs).

Thus, if we agree to represent unity by a length of 3 cms. then the number 3 would be represented by a line 9 cms. long, and a line of length 10·5 cms. would represent the number 3·5.

In all cases of the representation of physical quantities by lengths, the scale of the representation, *i.e.* the length representing the unit quantity, must be given either directly or by implication.

Masses to Scale.

EXAMPLE. *To construct a scale of masses so that the mass corresponding to any length, and the length corresponding to any mass, can be read off at once.*

The given line u (Fig. 1) represents 1 lb. mass. Transfer this length to your drawing paper, by pricking through with needle points or by the aid of dividers (having fine adjustment), and mark the end points O (left) and 1 (right). Mark off on this line produced, lengths giving 2, 3, 4, ... 10 lbs., as follows:

In the figure $O1$ is (intentionally) not the same length as u.

(i) With dividers accurately adjusted to the length u, and with the right-hand point as centre (marked 1 in figure) describe a semicircle clockwise, pricking a slight mark at the point (marked 2 in figure) where the semicircle cuts the line. With 2 as centre describe a semicircle contraclockwise, pricking through at 3, and so on by alternate clock- and contraclockwise half

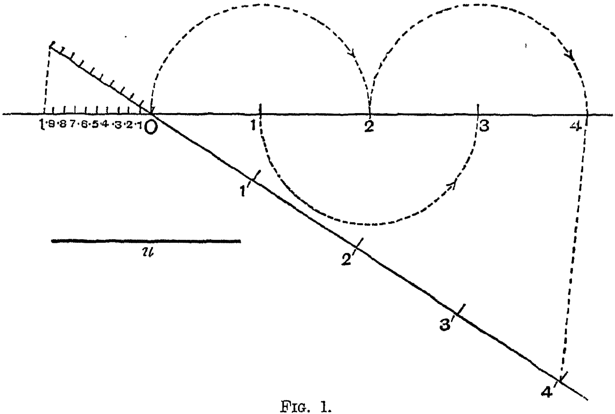


FIG. 1.

revolutions, pricking through points 4, 5,... 10. With a properly adjusted straight edge and set square draw fine sharp, short lines perpendicular to $O1$ through the points marked.

(ii) Draw a straight line through O making some angle between 20° and 60° with $O1$, and mark off from O inches $1'$, $2'$, $3'$, ... $10'$ along it.

Adjust a straight edge and set square with one edge of the latter passing through 1 and $1'$, so that when the set square is moved parallel to itself along the straight edge to $10'$ it still intersects $O1$ produced. Mark the points on $O1$ produced when the set square passes through $2'$, $3'$, ... $10'$ by short, sharp, fine lines.

These points so determined should coincide with the points already marked on $O1$ produced; why?

We have now $O1$ representing 1 lb. mass, and $O7$ a mass of 7 lbs., etc.

(iii) To obtain, by method (ii), the division marks perpendicular to $O1$.

Draw a fresh straight line and mark off $O1 = u$ on it. Place the inch scale and set square so that $1'\,1$ is perpendicular to $O1$ when the scale edge passes through O, and mark off the points 2, 3, ... as before.

EXAMPLE. *To find the length which represents* 3·7 *lbs.*

(*a*) With a scale, adjusted at O as before, mark the point 3·7″ from O along the scale, and with set square adjusted at this point parallel to $3\,3'$, mark a point on $O1$ produced 3·7, then the length from O to 3·7 represents 3·7 lbs.

(*b*) Produce $1O$ and $1'O$ backwards through O, and with an inch scale mark tenths of inches along the latter up to 1 inch, and from these points draw parallels to $3\,3'$ cutting $O1$ produced in points marked 0·1, 0·2, 0·3, ... 0·9. Then the distance between 0·7 and 3 represents 3·7 lbs.

The final result in (*b*) is a scale of masses from which the length corresponding to any mass between O and 11 lbs. or the mass corresponding to any length may be found.

(1) With squared paper or straight edge ruled in mms. or 2 mms. find the number represented by the line a, if (i) 1 cm. (ii) 2 cms. represents unity.

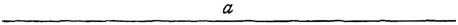


Fig. 2.

(2) Make a scale for numbers from 0 to 10 on squared paper, the length representing unity being u.

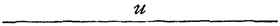


Fig. 3.

(3) On a plan of a house ¼ inch represents 3 feet. Draw a scale giving feet and ½ feet. What length represents 7 ft. 6 in., and what length is represented by 3·2 ins., and by the line a?

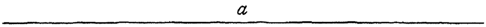
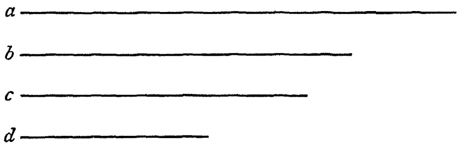


Fig. 4.

(4) The areas of certain fields are represented by lengths to the scale of 6 cms. to an acre. Draw a scale giving 1 to 5 acres, tenths of an acre and hundredths of an acre. Read off from your scale the area represented by 17·3 cms. and the length which represents 4·25 acres.

Addition.

Example. *The lines* a, b, c, d, *represent numbers to the scale of* $\frac{1}{2}$ *an inch to unity. Find the sum of the numbers.*



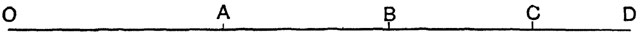


Fig. 5.

Take a strip of paper with a straight edge and apply in turn to the lines, marking with a fine sharp line the beginnings and ends of the segments so that the segment OA is equal to a, AB is equal to b and so on.

The edge, then, is marked $OABCD$ as in Fig. 5.

In Fig. 5, OD is one-half the true length.

Measure OD in half inches (or in inches and multiply mentally by 2), this number of half inches is the required sum.

Notice that the order of addition is immaterial.

(5) A scale pan is suspended from the hook of a spring balance, and it is loaded with small shot. The shot is put in by means of a small scoop. The weight of shot added each time is given by the lines a, b, c, d, e, and the line u represents 1 oz.

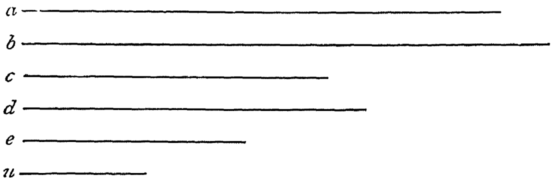


Fig. 6.

Find graphically the reading of the spring balance at each addition to the load.

(Add the lengths as above and then draw the u scale along the straight edge.)

(6) A weight of shot given by the line in Fig. 7 is taken out of the scale pan; what is the reading of the balance?

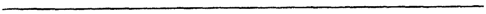


Fig. 7.

(7) What is the perimeter of the room of which the accompanying figure is the plan, drawn to a scale of 0·6″ to 11 ft.

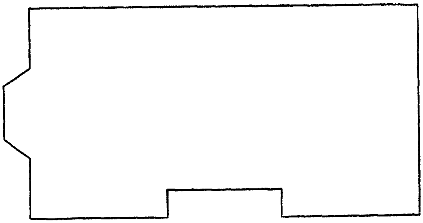


Fig. 8.

Subtraction.

Example. *The lines* a, b, c, d, (Fig. 9) *represent numbers to the scale of* 1·5 *cms. to unity. Find the difference between the second number and the sum of the rest.*

Add the lengths $a+c+d$ as before and obtain OD on the straight edge, cut off from D to the left $DB=b$, then OB is the length representing the required number. (In Fig. 9, *OA*, ... are half their true lengths.) Read off the length of *OB* on the 1·5 cm. scale.

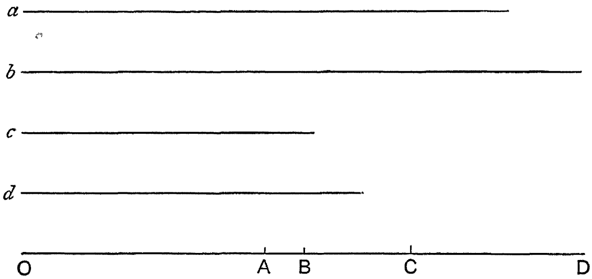


FIG. 9.

Notice that since addition is performed as a continuous process by adding lengths from left to right, subtraction must be performed by setting off distances from right to left, if we wish to measure our result from *O*.

EXAMPLE. *Required the number equal to the sum of the first and third numbers minus the sum of the second and fourth.*

Mark, as before, on a straight edge, $OA=a$, $AC=c$, then to the left, $CD=d$ and $DB=b$. The point *B* comes to the left of the starting point *O* (the origin), and the length *OB*, measured to the left instead of the right, corresponds to the fact that the required difference is negative. Measure *OB* on the proper scale and prefix a negative sign to the number.

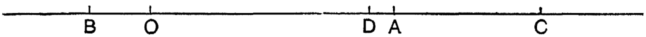


FIG. 10.

If distances to the right of *O* represent positive numbers, distances to the left must represent negative numbers.

Scale of Numbers. Such a line as *BODAC* (Fig. 10) when produced both ways represents numbers to the scale of 1·5 cms. to unity. Every distance to the right of *O* represents some definite positive number, every distance to the left represents some definite negative number; conversely, to every number corresponds a definite point in the line.

(8) Find the sum of the numbers represented by a, b, c, d, the scale being 0·4 inches to unity.

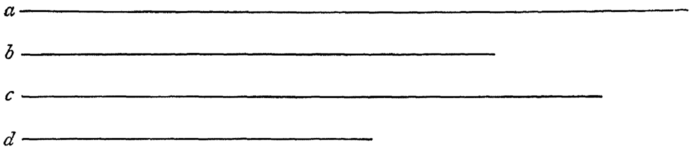


FIG. 11.

(9) Find the algebraic sum corresponding to $a+b-c+d$.

(10) Find the algebraic sum corresponding to $a-b-c-d$.

(11) Shew by actual measurement that

$$a+b-c+d=a+d-c+b,$$

and that

$$a-b-c-d=-b-c+a-d.$$

Similar Triangles. The construction on page 3 depended for its validity on a property of similar triangles, viz. **the ratios of the sides, taken in order, about the equal angles are equal.** For triangles we can always ensure similarity by making them equiangular. Generally, one figure is similar to another when it is a copy of the second drawn to the same or a different scale (in the first case the figures are congruent, *i.e.* identically equal).

(12) Draw any triangle ABC and by the aid of the right angle of a set square and a straight edge construct another triangle $A_1B_1C_1$, whose sides are perpendicular to those of the first. Scale the sides and calculate the ratios

$$\frac{AB}{A_1B_1}, \frac{BC}{B_1C_1}, \frac{CA}{C_1A_1}.$$

(13) By aid of the 30° set square construct $A_2B_2C_2$ such that A_2B_2 is turned clockwise through 30° from AB, and so on for the other sides. Verify again that the triangles are similar.

A property of similar triangles often useful in graphical work is that the ratio of their altitudes is equal to that of their bases.

(14) Verify this fact for the triangles drawn in the last exercise.

The graphical constructions for multiplication, division, etc., depend on these properties of similar triangles. It should be borne in mind that if the only object is to obtain the product, quotient, root, or power of numbers, the graphical constructions are but poor substitutes for abridged arithmetic, the slide rule and logarithms; it is only when in the course of other graphical work it is found necessary to obtain, say, the product of two numbers represented by lengths that the full advantage of the methods becomes apparent.

Notation. To avoid circumlocutions and the constant repetition of 'the number represented by the length,' it is convenient to use small letters a, b, c, ... for the lengths of lines, the numbers represented by these lines being denoted by the corresponding capitals A, B, C, ... When the lengths a, b, ... are set off from an origin O or U, they will be lettered OA, OB, ... or UA, UB, ... The line representing unity is designated by u, unless some measure in inches or centimetres is given.

Multiplication.

EXAMPLE. *The lengths* a (5·98 *cms.*) *and* b (8·84 *cms.*) *represent numbers to the scale* 1·5″ *to unity. Find,* (i) *the length which gives the product of the numbers,* (ii) *the product itself.*

(i) Draw any two intersecting lines (Fig. 12). From the point O of intersection set off $OU = 1{\cdot}5''$ and $OB = b$ along one, and on the other set off $OA = a$. Place a set square along AU and move it parallel to itself until it passes through B, mark C on OA where the set square cuts it.

OC is the required length, measure it by setting off the u scale along OC and obtain the product.

Proof. $\dfrac{OC}{OA} = \dfrac{OB}{OU}$, and, using A, B, C and U as numbers,

$$\frac{C}{A} = \frac{B}{1}, \quad \text{or} \quad C = A \, . \, B.$$

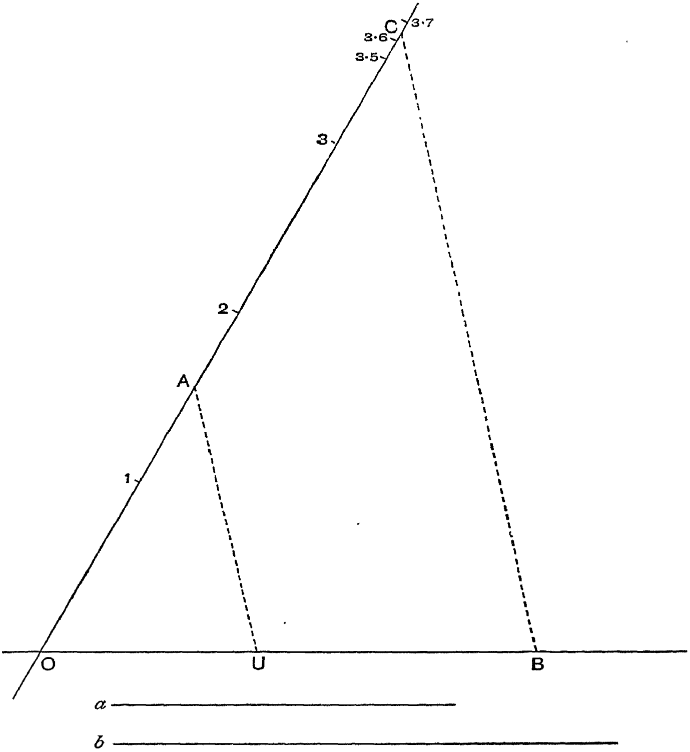

C
3·7
3·6
3·5
3
2
A
1
O
U
B
a
b

FIG. 12.

(ii) The construction given involves the transfer of lengths from the given to the drawn intersecting lines. If the lines a and b be already on the drawing paper this can be avoided.

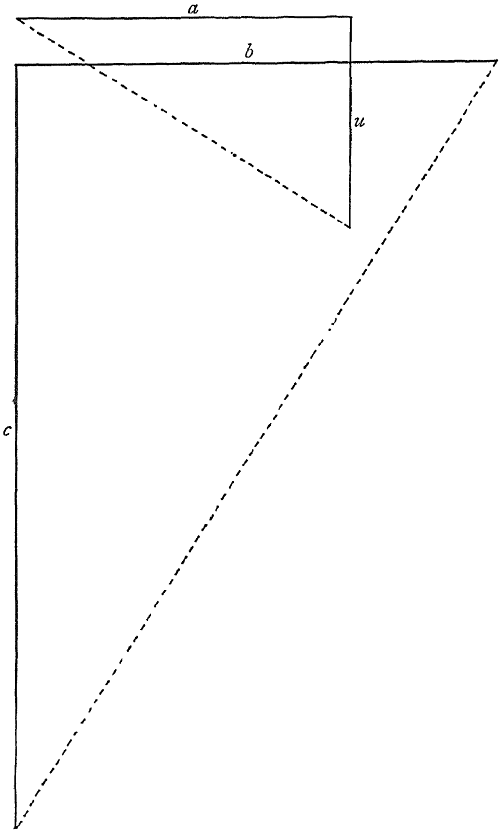


FIG. 13.

From one extremity of a draw u perpendicular to a (Fig. 13). From one extremity of b draw a line c perpendicular to b, and from the other a line perpendicular to the hypotenuse of the first right angle constructed. Then two similar triangles have been drawn, the sides of the one being perpendicular to those of the other.

Construct the u scale along c and measure c on that scale; it gives the product required.

Proof. From Fig. 13 $\frac{c}{b}=\frac{a}{u}$, or $\frac{C}{B}=\frac{A}{1}$, and $\therefore$ $C=A.B$.

If the lines a and b are not parallel, draw u at one end of a parallel to b, and complete the triangle; then from the extremities of b draw lines parallel to a and to the third side of the first triangle.

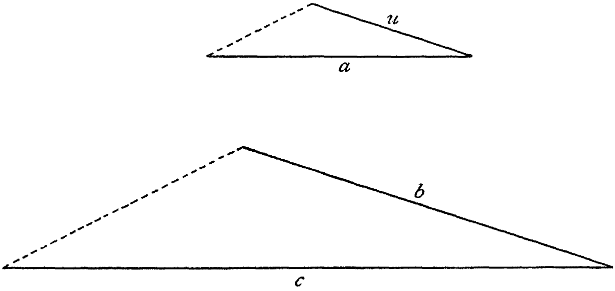


Fig. 14.

The two triangles are similar (since they are drawn equiangular) and the line c giving the product is parallel to a.

For $\frac{a}{u}=\frac{c}{b}$, or $ab=cu$, and $A.B=C$.

Measure c on the u scale and compare with the previous results.

*(15) Vary the construction in the case where a is parallel to b, by drawing u at an angle of 60° with a.

(16) Draw two lines of lengths 7·2 and 3·9 cms. Let these represent numbers to the scale of 0·7″ to unity. Find the product by the methods given. If unity be represented by 1·1″, find the product of the new numbers represented by the old lengths 7·2 and 3·9 cms.

Multiplication on Squared Paper.

EXAMPLE. *If* u = 2″, a = 8·38 *cms*, b = 6·82 *cms., find the product* A × B.

Take a sheet of ordinary squared paper (inches and tenths). Mark off as indicated in Fig. 15, $OA = a$, $OU = u$, and $UB = b$. Join OB and produce. Read off at once by the aid of the ruled lines the length of AC (AC being parallel to UB) on the 2 inch scale; it measures the product of A and B.

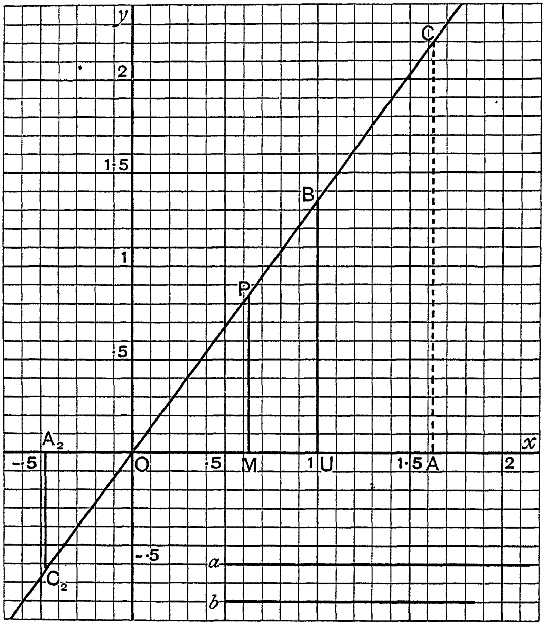


FIG. 15.

The side of each small square represents the number 0·05. With a little practice a fifth of this, or the number 0·01, can be estimated by the eye. To render the figure clearer in its reduced size, the side of the smallest square shewn represents 0·1 and not 0·05.

Mark the points on *OA* and *OA* produced corresponding to the numbers, 0·5, 1, 1·5, 2, 2·5, 3, 3·5 and 4, and to -1 and -2. On the line through *O* perpendicular to *OA* mark off the points corresponding to the same numbers.

This method is exceedingly convenient when more than one number has to be multiplied by the same factor. Any other number being given by a length a_1, we set off $OA_1 = a_1$, and then read off the length of the corresponding perpendicular A_1C_1, which is the product $A_1 \times B$.

(17) Read off the products of *B* and 2·7, 3·1 and 0·6.

(18) What number is represented by *b*? Read off the products of this number and 1·4, 2·3, 2·38 and compare the results with those obtained by actual multiplication.

(19) Multiply graphically 1·75 by 1·16, 2·35, 4·64, 3·88 and 5·26, using a scale of 2″ to unity.

(20) Multiply graphically 0·18 by 5·6, 2·4, 7·8, 6·9, using a scale of 1″ to unity horizontally, and 10″ to unity vertically. Read the products off on the vertical scale.

Equation to a Straight Line. Let any distance *OM* along the line *OA* (Fig. 15) be *x*, and the corresponding perpendicular distance *PM* be *y*. Then, wherever *M* may be along *OA* or *OA* produced,

$$\begin{aligned}\frac{y}{x} &= \frac{b}{u} \text{ always}\\ &= 1{\cdot}34\\ &= \text{tangent of the angle } OP \text{ makes with } OM\\ &\qquad\text{(called the slope of the line).}\end{aligned}$$

The equation $y = 1{\cdot}34x$ is called the equation to the straight line *OP*. *x* and *y* are called the **coordinates** of the point *P*, and the lines *OA* and its perpendicular through *O* (lettered *Ox* and *Oy* in Fig. 15) are called the axes of coordinates. *PM* or *y* is called the **ordinate**, *OM* or *x* the **abscissa** of the point *P*. *O* is the origin.

The **straight line** is called the **graph** of the corresponding equation $y = 1{\cdot}34x$, and any number of points on it could have been obtained by giving *x* values 1, 2, 3, ... , and calculating the corresponding values of *y* and marking* the points having these coordinates, or at once by drawing a straight line through O at a slope $= 1{\cdot}34$.

---

*The points should be marked with a ×, the two limbs of which must be fine and sharp and intersect at the point.

Take any point A_2 on OA produced to the left, then OA_2 represents a negative number (A_2). Read off what this number is. Produce OB backwards through the origin and read off the length on the 2″ scale of the perpendicular line A_2C_2. This number is the product $B \times A_2$. How does the figure shew that the product is negative?

(21) Find the product of B and $-0{\cdot}8$, $-1{\cdot}5$, $-2{\cdot}3$.

(22) Find the product of $-B$ and $0{\cdot}8$, $1{\cdot}5$, $2{\cdot}3$ directly from a figure, (set off b downwards).

Since distances upwards along Oy represent positive numbers, distances downwards along Oy produced must be considered negative.

Note also that since $\dfrac{-y}{-x} = \dfrac{y}{x}$, the equation $\dfrac{y}{x} = 1{\cdot}34$ represents the line POC_2 produced indefinitely both ways.

(23) Draw the straight lines whose equations are

$$\begin{aligned} y &= 3x \\ y &= x \\ y &= 1{\cdot}5x \\ y &= 0{\cdot}5x \\ y &= 0{\cdot}1x \end{aligned}$$

i.e. draw lines through the origin the tangent of whose angles with the axis of x are 3, 1, ...

(24) In the third equation of the last exercise suppose x to have values -2, -1, 0, 1, 2, 3, in turn; calculate the corresponding values of y, and shew that the points having these numbers as coordinates lie on the line already drawn.

Notice that $y = 0{\cdot}1x$, $y = 0{\cdot}01x$, $y = 0{\cdot}001x$, $y = 0{\cdot}0001x$ are successively nearer to the axis of x, and hence $y = 0 . x$ or $y = 0$ must be the axis $0x$ itself. Similarly $x = 0$ must be the axis of y.

(25) The lines a and b represent numbers to the scale u to unity.

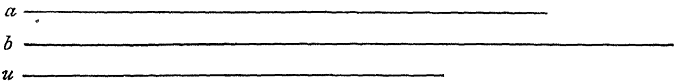


FIG. 16.

Find by method (ii) the product of the corresponding numbers, constructing the u scale along c.

(26) Find the product by method (i).

(27) Using paper divided into mms. or 2 mms., find the product of 1·74 and 0·82, 1·67, 2·31, 0·63, $-1·31$ and $-2·36$.

Different Scales. It is not necessary to use the same scale horizontally and vertically. Suppose we wish to multiply 0·27 by 6·6; it would be better to represent the first number to a scale 10 inches to unity, and the latter to a scale 1 inch to unity, than to take the same scale and have lines differing greatly in length. If the vertical scale be chosen as 10″ to unity, then the product must be read on that scale, for

$$\frac{CA}{BU}=\frac{OA}{OU},$$

and if OA and OU be measured on the same scale so must CA and BU.

Perform this multiplication graphically; mark points U where $OU=1'$, and A where $OA=6·6''$; set up, perpendicular to OA, $UB=2·7''$, *i.e.* 0·27 of ten inches; join OB and produce and read off on the ten inch scale the length of AC.

(28) Multiply 0·037 by 8·1, 7·3, 5·6, 2·9 and 10·3.

Division. If $C=A.B$ then $\frac{A}{1}=\frac{C}{B}$; and, therefore, to find the quotient $\frac{C}{B}$, we have to make but a slight modification in our construction for multiplication.

EXAMPLE. a *and* b *represent numbers to the scale* 2 *cms. to unity. Find the line representing the quotient* $\frac{\mathrm{A}}{\mathrm{B}}$ *and the number itself.* a = 3·92″, b = 1·34″.

Notice that if we do not wish to find the line representing the quotient, but only the number itself, we may take any length whatsoever to represent unity. This follows from the fact that $\frac{a}{b}=\frac{A}{B}$ whatever the scale may be.

(i) Draw any two intersecting lines (Fig. 17), from the point O of intersection set off $OU=2$ cms. along one, and $OA=a$ and

$OB = b$ along the other. Mark the point C on OU produced where the line through A parallel to BU cuts it. Then OC is the required length c; read the length on the 2 cm. scale and the required quotient 2·93 is obtained.

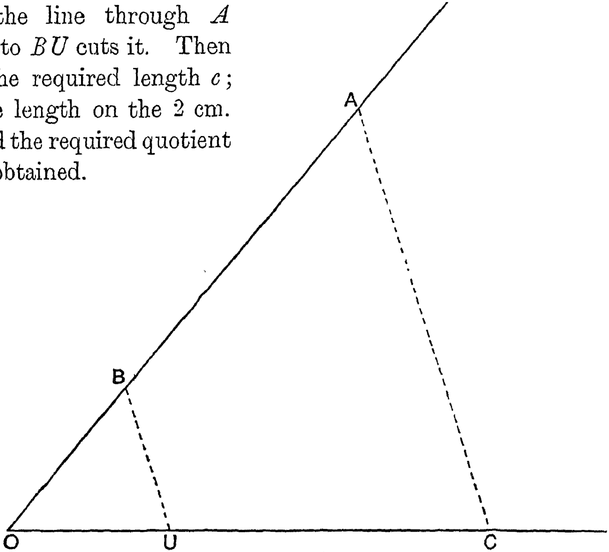


FIG. 17.

(ii) Draw u (Fig. 18) perpendicular to b at its extremity. On a construct a triangle similar to the one on b having its sides

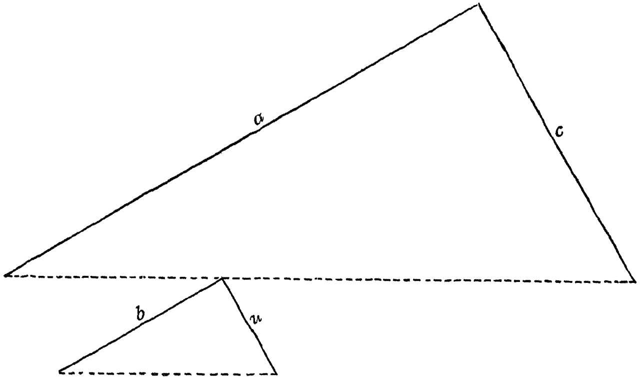


FIG. 18.

parallel to those of the first. Then c being the side corresponding to u we have

$$\frac{a}{c}=\frac{b}{u},\ \textit{i.e.}\ \frac{A}{C}=\frac{B}{1}\ \text{or}\ C=\frac{A}{B}.$$

(iii) On squared paper (mm.) take two axes at right angles. Set off $OU=2$ cms. (Fig. 19) and $OB=b$ along the axis Ox (the axis of x), $BA=a$ parallel to the axis Oy (the axis of y). Finally, UC, perpendicular to Ox, cutting OA in C, is the length representing the quotient. Read this length off on the 2 cm. scale and obtain the quotient C.

(29) Using squared paper find the quotient corresponding to

$$\frac{a}{d},\ \frac{b}{d}\ \text{and}\ \frac{c}{d}\ \text{(Fig. 20),}$$

where 5 cms. represents unity.

(30) Find by aid of squared paper the quotients of 5·6, 4·7, 2·8, 1·8, −2·6 and −1·5 by 2·6. (The negative a's must be set off downwards.)

(31) What are the equations to the sloping lines used in the two previous exercises?

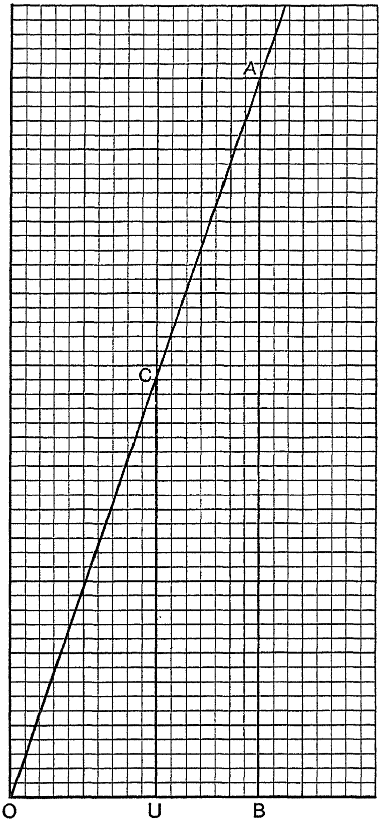


FIG. 19.

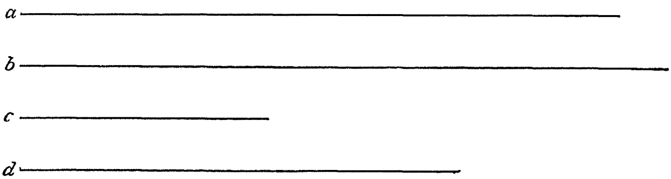


FIG. 20.

(32) Find by method (ii) the quotient representing $\frac{a}{b}$ where u represents

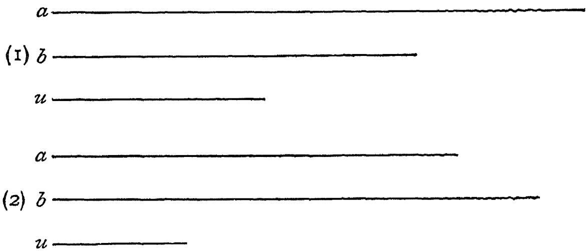


FIG. 21.

unity. Change the length of u to 2″ and see that the quotient is the same, but that the line representing it is altered.

(33) Find by direct graphical construction the quotients of 2·8, 5·7, 4·5, 3·7, −2·1 and −1·8 by −3·3.

Combined Multiplication and Division.

EXAMPLE. a, b *and* c *represent three numbers and* u *represents unity. Find the line which represents* $\frac{A . B}{C}$ *and the number itself.*

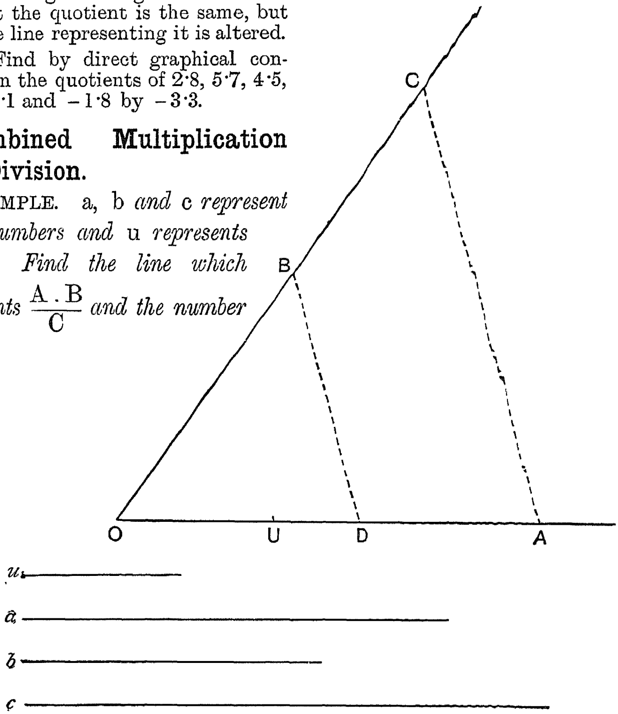


FIG. 22.

(i) Draw two intersecting lines and set off along one $OU=u$ (Fig. 22), $OA=a$, and along the other $OB=b$ and $OC=c$. Mark the point D on OA where BD, parallel to AC, cuts it, then OD is the required length.

Proof. $$\frac{OD}{OA}=\frac{OB}{OC} \text{ and } D=\frac{A\,.\,B}{C}.$$

Construct the u scale along OD and read off the number D.

(ii) Multiply the ratio $\frac{A}{C}$ by a set of numbers B_1, B_2, B_3, …, unity being represented by 0·5 inches.

On squared paper mark two axes Ox and Oy (Fig. 23); set off along Ox the distance c and draw $CA=a$ parallel to Oy.

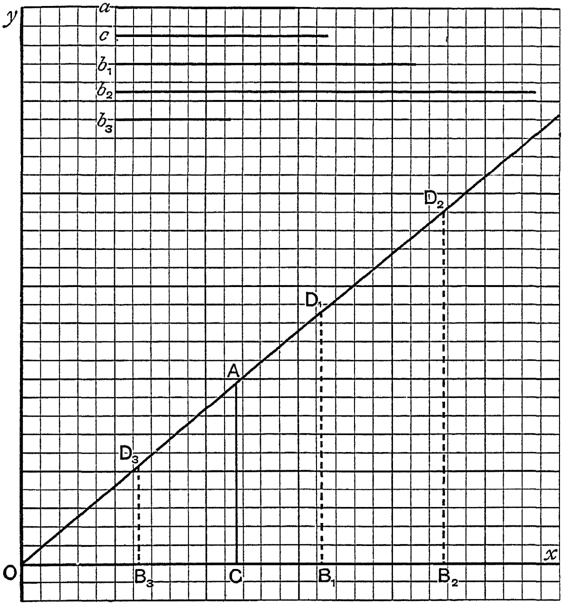


FIG. 23.

Join OA and mark the points on it where the y ordinates through B_1, B_2, B_3, …, cut it, viz. D_1, D_2, D_3. Then B_1D_1, B_2D_2, B_3D_3 are the required lengths. Read off the corresponding numbers.

(34) Draw four lines of lengths 7·8, 2·6 and 3·1 cms. and 0·4 inch. If the last represents unity, find the product of the numbers represented by the third and the ratio of the first to the second.

Continued Multiplication.

EXAMPLE 1. *Find a line representing the product* A . B . C . D *where* u *represents unity, the numbers being given by the lines* a, b, c, d.

Set off along any line $OU=u$ (Fig. 24),

$$OA=a,$$

$$OB=b,$$

$$OC=c,$$

and along an intersecting line $OD=d$.

Mark X_1 where AX_1, parallel to UD, cuts OD produced.
,, X_2 ,, BX_2 ,, UX_1 ,, OD.
,, X_3 ,, CX_3 ,, UX_2 ,, OD produced.

Then OX_3 is the length required. Measure OX_3 on the u scale and obtain the product $A \,.\, B \,.\, C \,.\, D$.

Proof.

$$\frac{OX_1}{OD}=\frac{OA}{OU},$$

$$\frac{OX_2}{OX_1}=\frac{OB}{OU},$$

$$\frac{OX_3}{OX_2}=\frac{OC}{OU}.$$

Multiply these ratios together and obtain

$$\frac{OX_1 \,.\, OX_2 \,.\, OX_3}{OD \,.\, OX_1 \,.\, OX_2}$$

$$=\frac{OA \,.\, OB \,.\, OC}{OU^3}$$

$$\text{or } X_3=A \,.\, B \,.\, C \,.\, D.$$

EXAMPLE 2. *The scale being* 1 *inch to unity, find, on squared paper, the continued product of the numbers represented by* a, b, c *and* d.

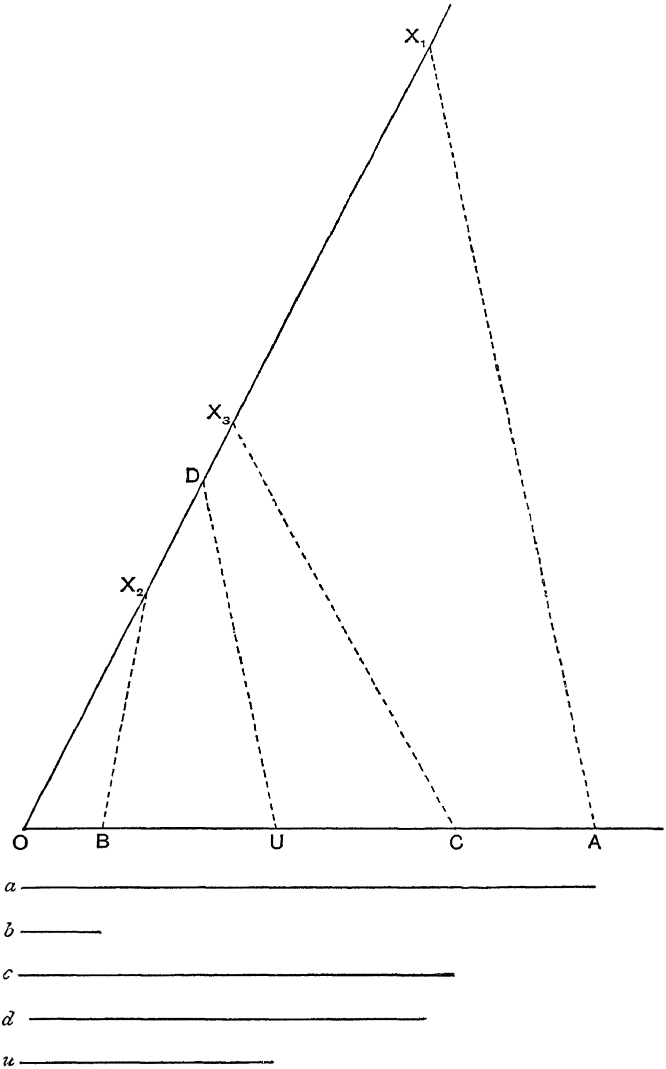

X₁
X₃
D
X₂
O
B
U
C
A
a
b
c
d
u

FIG. 24.

Mark positions for U, A, C and D along Ox (Fig 25), and set up $UB_1 = b$, parallel to the axis of y.

Join[1] OB_1 and produce. Mark the point A_1 on OB_1 where the ordinate at A cuts it. Mark the point A_2 where A_1A_2, parallel to Ox, cuts UB_1. Join[1] OA_2 and produce and mark on it C_2 where the ordinate at C cuts it. Join[1] OC_2 and mark C_3 on UB where C_2C_3, parallel to Ox, cuts it. If OC_3 be joined, the ordinate at D would not intersect it on the paper, so mark C_3' on the ordinate at $2U$ and join OC_3', marking D_3 where the ordinate at D cuts it. Then $2DD_3$ gives the product required. Measure this on the u scale and write down the product.

Proof. $$\frac{OA}{AA_1} = \frac{OU}{UB_1}, \quad \frac{OC}{CC_2} = \frac{OU}{UA_2}, \quad \frac{OD}{DD_3} = \frac{2OU}{UC_3};$$

$$\therefore \frac{OA \,.\, OC \,.\, OD}{AA_1 \,.\, CC_2 \,.\, DD_3} = \frac{2OU^3}{UB_1 \,.\, UA_2 \,.\, UC_3}.$$

But $AA_1 = UA_2$, $CC_2 = UC_3$;

$$\therefore \frac{A \,.\, C \,.\, D}{DD_3} = \frac{2}{B};$$

$$\therefore A \,.\, B \,.\, C \,.\, D = 2DD_3.$$

Do the multiplication again, taking A, B, C, on Ox and UD_1 vertically. When the construction lines go off the paper use the $2u$ line instead of the u line.

Change of Scale. When the lengths a, b, c, etc., are long compared with u, or when there are many multiplications to be performed, the lines OC_2 OD_3 ... in (ii) become so steep that their intersections with the verticals CC_2 DD_3 ... will not be on the paper. Similarly in (i) X_1, X_2, X_3 ... get farther and farther along OD, and the lines joining them to U become more and more nearly parallel to OD.

When this is the case the triangles become **ill-conditioned.** To avoid this difficulty the scale must be changed. Thus, in (i) if OCX_3 becomes an ill-conditioned triangle in consequence of b being much larger than in Fig. 24, either halve OX_3, or double

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[1] The lines need not actually be drawn, it is sufficient to mark the points.

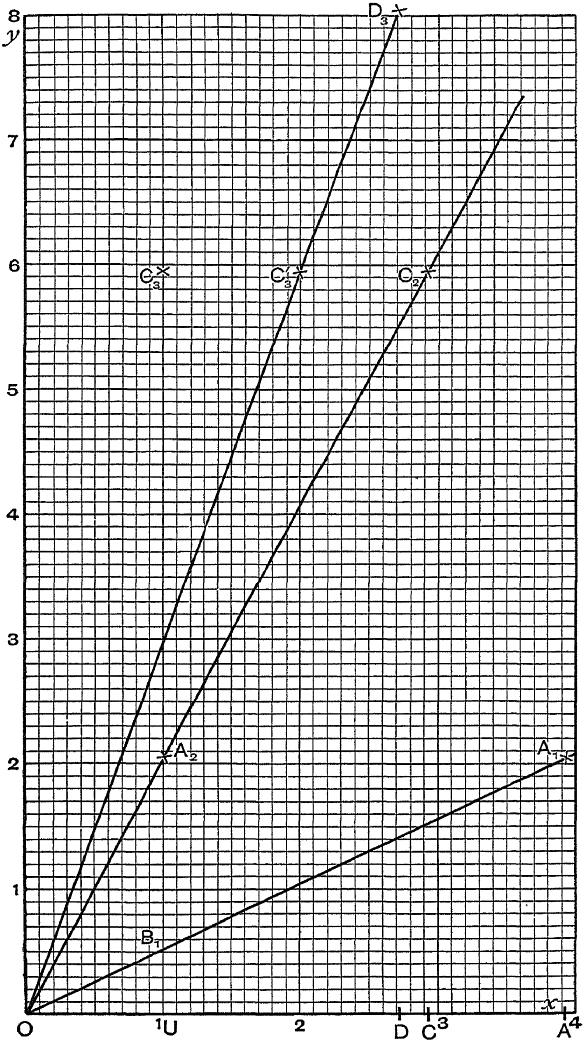

Fig. 25.

OU, and proceed as before; the resulting length OX_3' must now be measured on the scale of $\frac{1}{2}u$.

Should OCX_3' be still an ill-conditioned triangle, take $\frac{1}{4}$ of OX_3 or quadruple OU; if still ill-conditioned take $\frac{1}{10}$ of OX_3 or ten times OU, and read the answer on $\frac{1}{4}$ or $\frac{1}{10}$ of the u scale.

A similar change can be made in method (ii) if necessary.

(35) Find the value of $\frac{A \,.\, B}{C}$, where

$a = 5{\cdot}2''$,
$b = 2{\cdot}7''$,
$c = 1{\cdot}4''$,
$u = 4$ cms.

(36) Find, on squared paper (mm.), the value of $\frac{A}{C} \times B_1 \,.\, B_2 \,.\, B_3$, where

$u = 1''$,
$a = 5{\cdot}7$ cms.,
$c = 6{\cdot}8$ cms.,
$b_1 = 2{\cdot}3$ cms.,
$b_2 = 5{\cdot}6$ cms.,
$b_3 = 9{\cdot}2$ cms.

(37) Find the value of $\frac{6{\cdot}8 \times 2{\cdot}7 \times 1{\cdot}9 \times 3{\cdot}4}{5{\cdot}3}$ graphically.

(38) Find the value of $A \,.\, B \,.\, C \,.\, D \,.\, E$, where

$a = 6{\cdot}3$ cms.,
$b = 5{\cdot}1$ cms.,
$c = 2{\cdot}7$ cms.,
$d = 3{\cdot}8$ cms.,
$e = 5{\cdot}9$ cms.,
and $u = 2''$.

Continued Product of Ratios.

EXAMPLE. *Find the line representing to the scale* u *to unity the product* $\frac{\mathrm{A}}{\mathrm{B}} \cdot \frac{\mathrm{C}}{\mathrm{D}} \cdot \frac{\mathrm{E}}{\mathrm{F}}$, *the numbers being given by the lines*

$a = 9{\cdot}9$ cms.,
$b = 5{\cdot}52$ cms.,
$c = 6{\cdot}3$ cms.,
$d = 3{\cdot}46$ cms.,
$e = 10{\cdot}6$ cms.,
$f = 13{\cdot}05$ cms.,
$u = 1{\cdot}65$ inches.

Set off OA, OB, OC, OD, OE, OF from O along any convenient line and OU along an intersecting one (Fig. 26).

Mark on OU the points X_1, X_2, X_3, where AX_1 is parallel to BU, CX_2 is parallel to DX_1, EX_3 is parallel to FX_2.

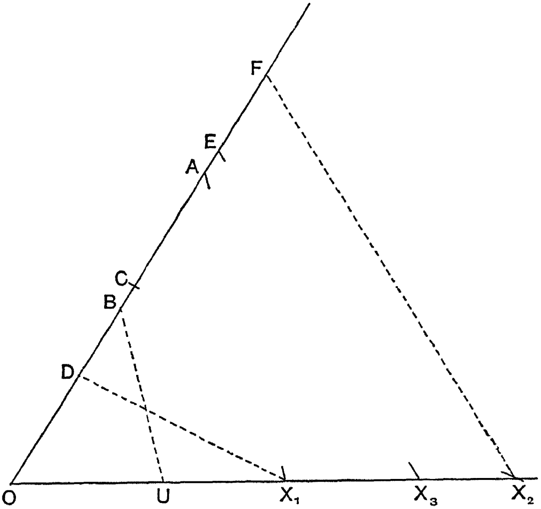

FIG. 26.

Then OX_3 gives the required product; measure this on the u scale.

Proof. $\dfrac{OX_1}{OU}=\dfrac{OA}{OB}$, $\dfrac{OX_2}{OX_1}=\dfrac{OC}{OD}$, $\dfrac{OX_3}{OX_2}=\dfrac{OE}{OF}$;

$$\therefore \frac{OX_2}{OU}=\frac{OA}{OD}\cdot\frac{OC}{OD}\cdot\frac{OE}{OF}, \text{ or } X_2=\frac{A}{B}\cdot\frac{C}{D}\cdot\frac{E}{F}.$$

(39) Draw lines of lengths 10·3, 7·8, 6·5, 4·3, 3·9, 2·7 cms., and find the continued product of the ratios of the first to the second, the third to the fourth, etc., if 0·7″ represents unity.

(40) Find the continued product of the first four numbers in Ex. 39 and the value of $\left(\dfrac{10{\cdot}3}{7{\cdot}8}\right)^3$.

Integral Powers (Positive and Negative). Since A^4 means $A\times A\times A\times A$ and A^{-3} means $\frac{1}{A^3}=\frac{1}{A\times A\times A}$, it is evident that the constructions already given cover the cases in which numbers have to be raised to positive or negative integral powers. It is, however, simpler to use the subjoined construction.

EXAMPLE. *Given* a (1·62″) *and the unit line* u (3·08 *cms.*) *to construct lines giving* A^2, A^3, A^4, ... *and* $\frac{1}{\mathrm{A}}, \frac{1}{\mathrm{A}^2}, \frac{1}{\mathrm{A}^3}, \ldots$

Draw any two lines intersecting at right angles. Set off along these $OU=u$ and $OA=A$.

Join UA, and draw AA_2, A_2A_3. A_3A_4 ... so that each line is perpendicular to the one drawn immediately before it, as in Fig. 27. Also, draw UB_1, B_1B_2, B_2B_3 ... where the lines are parallel to AA_2, A_2A_3, ...

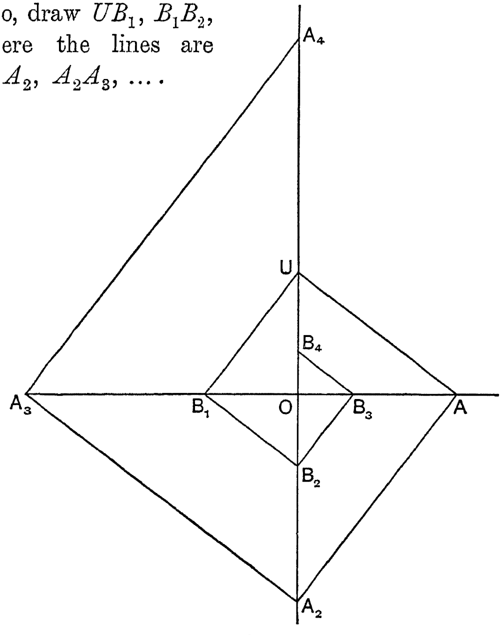

FIG. 27.

Construct the u scale along a straight-edged piece of paper.

Measure OA_2, OA_3, OA_4 ... on the u scale; they are A^2, A^3, A^4 ....

Measure OB_1, OB_2, OB_3 ... on the same scale; they are $\frac{1}{A}$, $\frac{1}{A^2}$, $\frac{1}{A^3}$ ....

Proof. All the triangles drawn are similar, and hence

$$\frac{OA_4}{OA_3}=\frac{OA_3}{OA_2}=\frac{OA_2}{OA}=\frac{OA}{OU}.$$

Multiplying the ratios, we get

$$\frac{OA_4}{OU}=\left(\frac{OA}{OU}\right)^4,$$

but OU represents unity,

$$\therefore\ A_4=A^4.$$

Similarly, $$\frac{OA_3}{OU}=\left(\frac{OA}{OU}\right)^3,\ \text{and}$$

$$\therefore\ A_3=A^3,\ \text{etc.}$$

For the reciprocals we have

$$\frac{OB_4}{OB_3}=\frac{OB_3}{OB_2}=\frac{OB_2}{OB_1}=\frac{OB_1}{OU}=\frac{OU}{OA}.$$

Multiplying together

$$\frac{OB_4}{OU}=\left(\frac{OU}{OA}\right)^4;$$

$$\therefore\ B_4=\frac{1}{A^4}.$$

Similarly, $$B_3=\frac{1}{A^3},\ \text{etc.}$$

For positive powers the construction stops at A_4, since A_5 would not be on the paper. Take, then, $\frac{1}{10}$ of OA_4 and proceed as before; then

$$\frac{OA_5}{\frac{1}{10}OA_4}=\frac{OA_4}{OA_3};$$

$$\therefore\ OA_5=\frac{1}{10}\frac{(OA_4)^2}{OA_3};$$

$$\therefore\ A_5=\tfrac{1}{10}A^5.$$

The succeeding intercepts must, therefore, be read on the $\frac{1}{10}$th u scale.

Again OB_5 is too small to measure accurately, take 10 OB_4 and read on the 10 u scale.

The points ... B_4, B_3, B_2, B_1, U, A, A_2, A_3, ... are points on a curve called the **equiangular spiral.** The intermediate points on this curve would give fractional and decimal powers, and would thus enable one to find the values of such expressions as $A^{\frac{3}{2}}$, $A^{\frac{2}{5}}$. It is not difficult to construct such a curve geometrically.

Square Roots.

EXAMPLE. *Find the square root of* A, *given* a *and* u.

Set off $OA=a$ (Fig. 28) and $OU=u$ in opposite senses along a straight line. On UA describe a semicircle UCA, and measure OC where OC is perpendicular to UA.

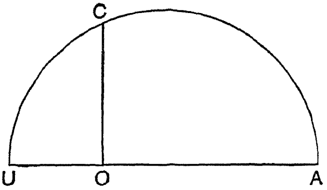

FIG. 28.

Then $$C=\sqrt{A}.$$

Proof. Since UA is a diameter of a circle and OC a semichord perpendicular to it;

$$OC^2=OU.OA;$$

$$\therefore\ C^2=A \text{ or } C=\sqrt{A}.$$

By repeating this process we can find rapidly $A^{\frac{1}{4}}$, $A^{\frac{1}{8}}$, ...

(41) Draw a line 6·5 cms. long. If unity be represented by 2″, find the powers of the number up to the 6th, and their reciprocals.

(42) Draw a line 6·5 cms. long. If unity be represented by 3″, find the square, cube and 4th power of the number, and their reciprocals.

(43) Find the square and 4th roots of the given numbers in (41) and (42).

Powers by Squared Paper.

EXAMPLE. *Construct a curve which gives by inspection the squares of all numbers, integral and decimal, from* -3 *to* $+3$.

Take two axes (Fig. 29) along the thick lines of the squared paper, the axis of x horizontally and the axis of y vertically. Mark the large divisions along the axis of x 0·5, 1, 1·5, 2, ..., and those along the axis of y 1, 2, 3, ... Draw OP through O, and the point $x=3$, $y=3$. In Fig. 29 part only of the squared paper and the curve is shewn.

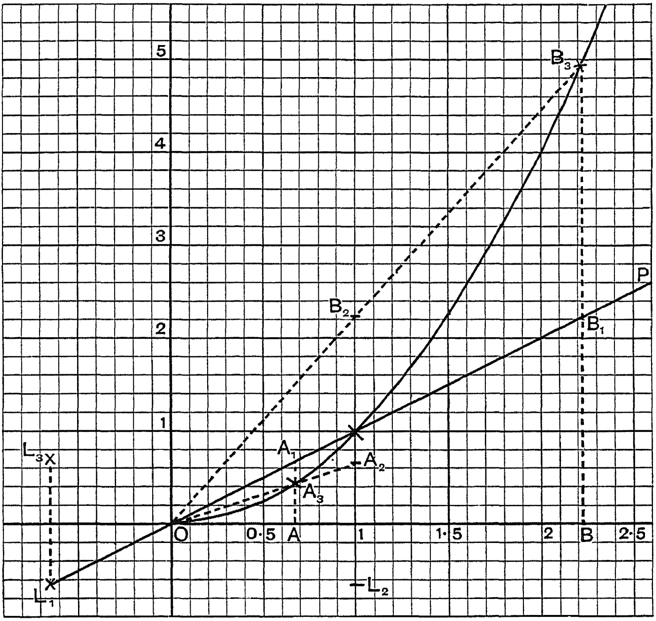

FIG. 29.

Mark any point A_1 on OP by a sharp short line perpendicular to OP.

From A_1 go horizontally to A_2 a point on the ordinate at 1, put a straight edge along OA_2, and mark the point A_3 where it cuts the ordinate through A_1.

Proceed similarly with points B_1, C_1, ..., along OP, taking at least twelve points. In Fig. 29, to save confusion, the construction lines for only 2 points A_3 and B_3 have been shewn. With a little care and practice the points A_3, B_3, ... can be marked accurately without actually drawing any construction lines except OP, the ruled lines in the paper being a sufficient guide for the eye.

Take points also on OP, produced backwards through the origin, such as L_1, and repeat the construction and find a number of points like L_3. Join all the points so obtained by a smooth curve drawn by freehand; see that it *is* a smooth curve by looking along it, and smooth down any humps and irregularities that appear on it.

The curve thus constructed is such that the ordinate for any point on it represents the square of the number given by the corresponding abscissa.

Proof. $0AA_3$ is similar to $01A_2$, and $0A = 2AA_1$ as lengths;

$$\therefore \frac{AA_3}{1A_2} = \frac{0A}{01} = \frac{AA_3}{AA_1}.$$

But AA_1 and $0A$ represent the same number, viz. A, though to a different scale; hence A_3 being the number given by AA_3, $A_3 = A^2$, *i.e.* AA_3 represents the square of the number A, the scale being one half that on which A is measured.

A similar proof holds for negative numbers, and the construction shews that the square of a negative number is positive.

(44) Read off from the curve as accurately as possible the squares of 0·52, 0·68, 0·84, 1·75, 1·98, 2·24, 2·5, 2·85.

(45) Read off the square roots of 0·55, 0·85, 2·54, 3·8, 3·6, 4·54.

(46) Find the squares of the numbers given by lengths, 1, 1·8, 2·3, 4·6, 5, 6·7 and 7 cm. if unity be represented by 2″.

(47) Find the square roots of the numbers given by the lengths, 2, 3·8, 4·7, 6·9, 8·5, 10·3, 11·1 cms. if 2″ represents unity.

Evidently the curve as drawn is not adapted for finding the squares of numbers much greater than 3. To find the squares of greater numbers, the scale of numbers along Oy must be made still smaller than that along Ox. Thus for numbers from 0 to

100 take 1 inch to represent 10 along *Ox*, but along *Oy* take 1 inch to represent 1000. The construction is almost exactly the same; the line *OP* joining *O* to the point for which $x = 100$, $y = 100$.

(48) Find graphically the squares of 2·7, 3·6, 7·7, 9·8, using a tenth scale along the y axis. Find the square roots of 87, 73, 60, 31, 20 and 12.

(49) Construct a curve giving $\frac{1}{7}$ of the squares of the numbers ranging from -4 to $+4$. (The line *OP* must now go through the point (7, 1).)

Equation to Graph. The curve just constructed (p. 30) is such that every ordinate like BB_3 represents a number which is the square of the number represented by the abscissa *OB*. If, then, y and x are these *numbers*, $y = x^2$, and since this equation holds for all points on the curve it is called the equation to the curve, and the curve is the graph of the equation.

It must be clearly understood that the equation $y = x^2$ is only true if, by y and x, we mean the *numbers* represented by the lines and not the actual *lengths* of the lines themselves.

If y and x denote the lengths representing the number then the equation $y = x^2$ is not true.

Let u be the unit length along *Oy*, and $2u$ the unit length along *Ox*, then, referring to Fig. 29, we have $OA = x$, $AA_3 = y$, and since

$$\frac{AA_3}{OA} = \frac{AA_1}{O1},$$

we get

$$\frac{y}{x} = \frac{\frac{1}{2}x}{2u}, \text{ or } y = \frac{1}{4u}x^2, \text{ or } \left(\frac{y}{u}\right) = \left(\frac{x}{2u}\right)^2.$$

The last form of the relation brings us back to the original equation; for $\frac{y}{u}$ is the number represented by the length y, and $\frac{x}{2u}$ is the number represented by the length x.

(50) If the scale along y had been 1″ to unity, and 5″ to unity along x, what would have been the equation connecting the lengths x and y of the coordinates of any point on the curve, and what would be the relation between corresponding numbers? What length would represent the square of the number 3?

(51) If a be the unit of length along x, and b that along y, what is the equation connecting the lengths x and y? If $a = 2·3″$ and $b = 1·5″$, what are the lengths representing the squares of 1 and 3?

Another construction for the curve $y = x^2$ follows from the geometrical method explained on p. 26.

Take two axes on squared paper (Fig. 30); let unity be represented by 1″ along Ox and $\frac{1}{2}''$ along Oy.

Take any point A on Ox and by the aid of set squares draw AA_1 perpendicular to the line joining A and -4 (on Oy).

Mark on the ordinate at A the point A_2 where A_1A_2, parallel to Ox, cuts it.

Repeat this construction for a number of points like A, and join the points like A_2 by a smooth curve; this is the curve of squares.

Proof. As on p. 28,

$$OA^2 = OA_1 \times O(-4).$$

Taking all measurements in inches, we have, if $OA = x$ and $OA_1 = y$,

$$x^2 = 2y$$

(x and y being in inches).

Hence, if the axis of y be marked $\frac{1}{2}''$ to unity, we have, *as numbers*,

$$y = x^2.$$

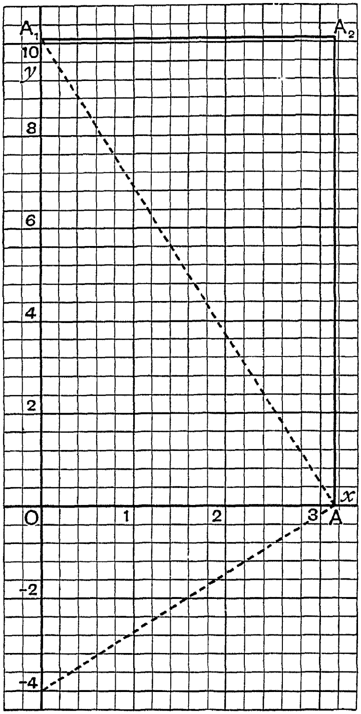

Fig. 30.

(52) Construct, by a similar method, a curve giving directly 1·6 times the square of numbers from -3 to $+3$.

Cubes and Cube Roots.

Example. *From the curve* $y = x^2$ *construct a curve giving the cubes of numbers from* -2 *to* $+2$.

First construct the curve of squares, the origin being in the centre of the squared paper, and the y scale $\frac{1}{2}$ that of the x.

Take any point A_1 (Fig. 31) on the curve, go horizontally to

A_2 on the vertical through 1, mark A_3 on AA_1 where OA_2 cuts it, then AA_3 gives the cube of A.

Proceed similarly with a number of other points like A_1. Join all the points similar to A_3 (for negative as well as positive x's); this is the curve giving the cubes.

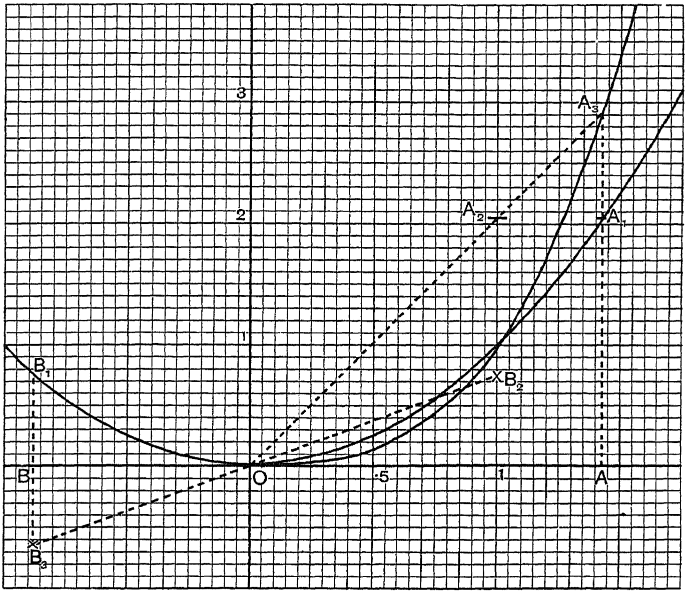

FIG. 31.

Proof.

$$AA_1 = OA^2 \text{ (as numbers)}$$

$$= 1A_2 \text{ and } \frac{1A_2}{O1} = \frac{AA_3}{OA};$$

$$\therefore\ OA^3 = AA_3 \text{ (as numbers)}.$$

If, then, y denote the number corresponding to any ordinate AA_2, and x the number for the corresponding abscissa, $y = x^3$, is the equation to the curve.

Notice that for a negative number x, y is negative.

(53) Find the cubes of 0·3, 0·5, 1·4, 1·7, 1·9, 2·1, 2·8 and 3·1, and the values of $\sqrt[3]{2}$, $\sqrt[3]{3{\cdot}5}$, $\sqrt[3]{8{\cdot}7}$.

(54) From the curve $y=x^3$ construct the curve giving the fourth powers of numbers from $-1{\cdot}8$ to $+1{\cdot}8$.

(55) Read off from the curve $y=x^4$, the values of

$$\sqrt[4]{2},\ \sqrt[4]{5},\ \sqrt{9},\ (1{\cdot}22)^4,\ (1{\cdot}85)^4.$$

(56) Draw a curve by the above construction giving $\frac{1}{4}$ of the cubes of numbers from 0 to 10. What is its equation?

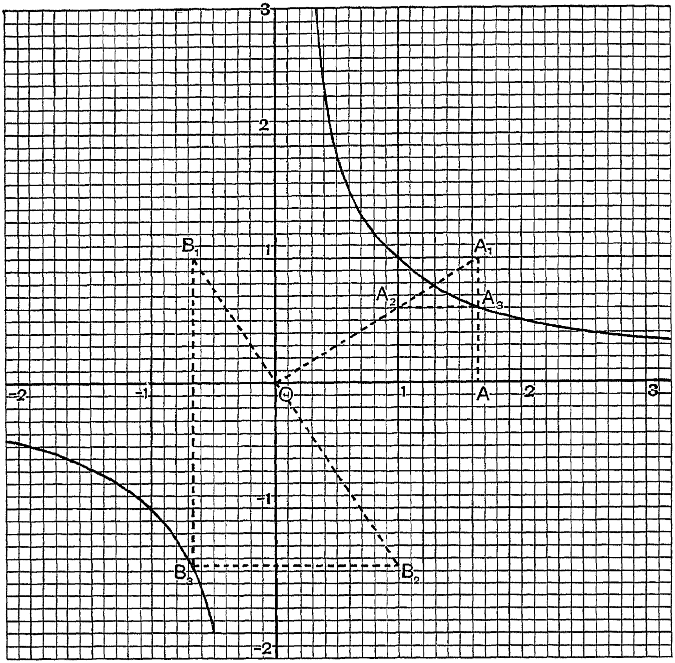

Fig. 32.

Curve giving Reciprocals. Let 1 inch represent unity. Take the origin at the centre of the squared paper. Mark any point A_1 (Fig. 32) on the unit line parallel to Ox, mark also A_2 where OA_1 cuts the unit line parallel to Oy. Go horizontally to

A_3, the point where A_2A_3 cuts the ordinate through A_1. Then AA_3 represents the reciprocal of the number A.

Repeat this process for a number of points like A_1 on the positive and the negative sides of Oy. Join all the points like A_3 by a smooth curve. This curve is such that the ordinate at any point A_3 gives the reciprocal of the corresponding abscissa number.

Proof. $AA_3 = 1A_2$ and $AA_1 = O1$.

Also
$$\frac{1A_2}{O1} = \frac{AA_1}{OA};$$

$$\therefore \; AA_3 = \frac{1}{OA} \text{ (as numbers).}$$

(57) Read off from the curve the reciprocals of 0·35, 0·75, 1·2, 1·85, 2·15, 2·75, 3·84, $-0·65$ and $-2·78$.

(58) Using a construction similar to that on page 32, draw the curve of reciprocals. [Take $OA = 1$ always, make $O(-4) = x$, then $OA_1 = y$.]

Equation to Curve of Reciprocals. Let y be any ordinate number, and x the corresponding abscissa number; then evidently from the construction

$$\frac{y}{1} = \frac{1}{x} \quad \text{or} \quad xy = 1,$$

which is the equation to the curve drawn.

Since
$$-x \,.\, -y = xy = 1,$$

we see that the two parts drawn by graphical construction are really branches of the same curve.

Notice also that for very big x's the y's are very small, and *vice versa*, hence as we travel along x in the positive sense the curve approaches nearer and nearer to the axis but never crosses it. Similarly, for very large negative x's the curve gets very near to the axis of x (negative side) but is below it.

(59) Construct the curve which gives $\frac{1}{4}$ of the reciprocals of numbers. What is its equation?

(60) Construct the curve giving 1·7 times the reciprocals of all numbers from 1 to 100. What is its equation?

Curve of Reciprocals Squared. *From the curve* $y=\frac{1}{x}$, *construct the curve* $y=\frac{1}{x^2}$, *giving the squares of the reciprocals of numbers.*

The construction is very similar to the last. Put a straight edge along OA_1 where A_1 is any point on the curve $xy=1$, mark A_2 where the straight edge cuts the ordinate at 1, go horizontally to A_3 on the ordinate AA_1, then AA_3 gives the required reciprocal squared.

How does the construction shew that $\left(-\frac{1}{x}\right)^2$ is positive ?

(61) Construct from the curve $y=\frac{1}{x}$ the curves

$$y=\frac{1}{2x^2} \text{ and } y=\frac{4}{x^2}.$$

(62) Construct from $y=\frac{1}{x^2}$ the curves

$$y=\frac{1}{x^3} \text{ and } y=\frac{3}{x^3}.$$

(63) From $y=x^3$ construct $y^2=x^3$, or $y=x^{\frac{3}{2}}$.

So far multiplication, division, etc., have referred to numbers, represented by lengths. On page 1 it was pointed out that a length may represent any other scalar quantity, the length representing the unit quantity being given.

Areas to Scale. The **product of two lengths** *a* and *b* is defined as **the area of a rectangle having** *a* **and** *b* **as adjacent sides**

The product of two unit lengths is unit area. To represent the product of two lengths by a line, we must first choose a line to represent unit area. This line may be the unit of length, or, if more convenient, some other length.

The methods, for finding the lines representing areas or volumes, are exactly the same as for multiplying numbers together; it is only the interpretation that is different.

EXAMPLE. *Represent the product of* a × b *by a line, unit area being represented by* u, *the unit of length.*

Set off $OU = u$ (Fig. 33), $OA = a$ along any line, and $UB = b$ perpendicular to it, then

$$AC \,.\, OU = OA \,.\, UB$$

or $$AC \,.\, u = a \,.\, b.$$

AC is the height of a rectangle having unit length as base. Measure this on the u scale; it gives the number of unit areas contained in $a \,.\, b$.

Note that although the same line u represents unit area and unit length, it is not correct to say that lengths and areas are represented to the same scale. They are different physical quantities, and all we can say is that the same length represents the same *number* of units of length as of units of area.

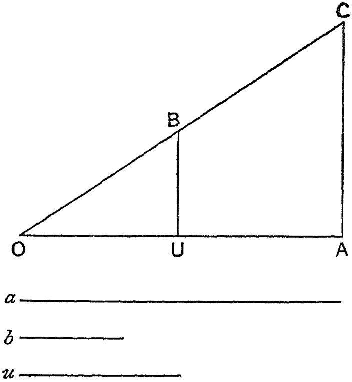

FIG. 33.

EXAMPLE. *Lines of lengths* 7 *and* 15 *cms. represent the sides of a rectangular room to the scale of* 1″ *to* 10′. *Find a line giving the floor area, when the unit area is*

(i) 10 sq. ft., (ii) sq. yds., (iii) 17 sq. ft.

(i) Draw, as in Fig. 33, $OU = 1''$, $OA = 15$ cms., $UB = 7$ cms.; produce OB to cut AC, the perpendicular at A to OA, in C. Then, as before, $AC \,.\, OU = OA \,.\, UB$.

This equation remains true whatever the scale on which we measure the lengths. If, then, we measure on the tenth inch scale, each tenth represents one ft. (for OU represents 10 ft.), and AC represents the height of a rectangle of base 10 ft. and area equal to the given floor, *i.e.* gives the floor area in 10 sq. ft.

(ii) Set off $OU = 0{\cdot}9''$ and measure AC in tenths of inches.

(iii) Set off $OU = 1{\cdot}7''$,, ,, ,,

Any Scale. Suppose the given lengths OA and UB represented lengths to the scale 1″ to x ft. and we want to find the area represented by $OA \times UB$ in sq. yds.

x being a number we can always set off a line representing that number of feet, and $\frac{9}{x}$ of this will be OU, and AC must be measured on the scale $\left(\frac{1}{x}\right)''$ to 1 sq. yd.

Thus, if $x = 7'$ we must divide 1″ into 7 equal parts (or if more convenient 10″ into 7 equal parts), 9 of these will equal OU. Make the construction as before, and measure AC on the scale of $\frac{1}{7}''$ to 1 sq. yd.

(64) Lines of lengths 2·3 and 4·7 inches represent, to the scale of 10 cms. to 7 ft., the sides of a rectangular room. Find by construction the floor area in sq. yds.

(65) Lines of lengths 1·82 and 3·65 inches represent the altitude and base of a rectangle to the scale of 1 inch to 350 cms., find geometrically the area in 100 sq. cms.

*(66) Find a line representing the volume of a rectangular box, in cb. ins., whose edges are 7, 15 and 17 cms. in length, unit volume being represented by 0·1 inch.

*(67) Find graphically the volume of a rectangular room whose dimensions are given by lines of 3·2, 5·3 and 6·7 cms., the scale being 1″ to 10′, (1) in cb. yds ; (2) in 10 cb. ft.

Work done. The work done in lifting a body vertically upwards is defined as the product of the weight of the body and the vertical distance moved through. If the weight be expressed in pounds and the distance in feet, the product is in **foot-pounds** (ft.-lbs.). The work done in lifting a 1 lb. weight vertically through 1 ft. is thus 1 ft.-lb., and is the unit of work. Obviously the work done in lifting 10 lbs. through 1 ft. is the same as that done in lifting 1 lb. through 10 ft. or 1 oz. through 160 ft.

EXAMPLE. w *represents the weight of a body to the scale* u *to a lb. weight,* s *represents the vertical distance moved through, to the scale* f *to a ft.; find graphically the work done in ft.-lbs.*

Notice that $u \times f$ is the area representing a ft.-lb., and $w \times s$ the area we wish to find in terms of $u \times f$.

We can most conveniently do this by finding the rectangle whose base is u or f and whose area is $w \times s$.

(i) Set off $OU=u$ (Fig. 34) and $OW=w$ along one axis, and $OF=f$ and $OS=s$ along an intersecting one.

Draw WX parallel to US and measure OX on the f scale.

This gives the work done in ft. lbs.

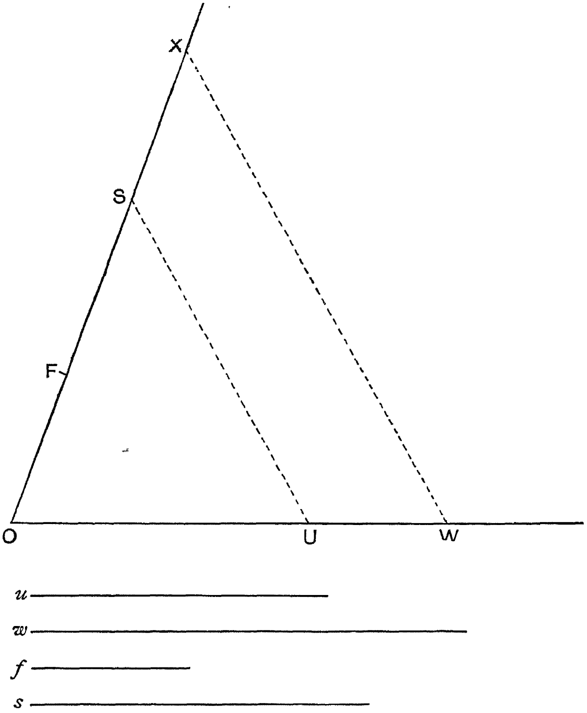

FIG. 34.

Proof. $$\frac{OX}{OS}=\frac{OW}{OU} \text{ or } OX \,.\, OU=OW \,.\, OS;$$

$$\therefore\ OX \,.\, u=w \,.\, s.$$

Hence OX is the altitude of a rectangle whose base is u and whose area is $w \,.\, s$; and therefore OX represents the vertical distance through which 1 lb. weight must be raised in order that the given work may be done. If, then, OX be measured on the f scale the number of units in OX will be the number of ft-lbs. represented by $w.s$.

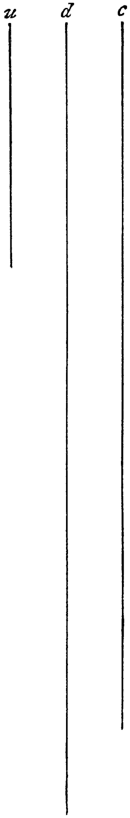

FIG. 36.

(ii) Set off $OF=f$ (Fig. 35) and $OW=w$ along any line, $OS=s$ along an intersecting line, and draw WX parallel to FS.

Then $$\frac{OW}{OX}=\frac{OF}{OS};$$

$$\therefore\ OF\,.\,OX=OW\,.\,OS;$$

i.e. $f\,.\,OX=w\,.\,s$.

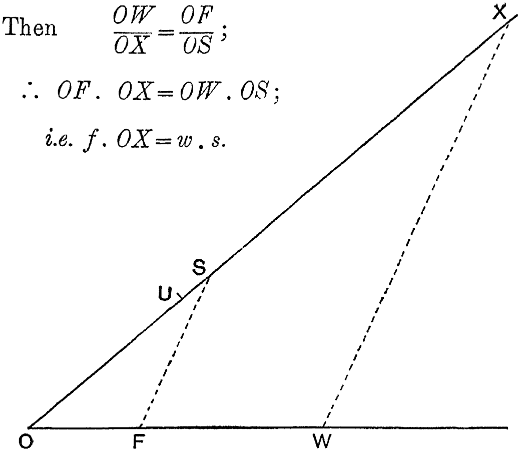

FIG. 35.

OX therefore measures the weight which, lifted vertically through 1 ft., requires an expenditure of work represented by $w.s$. Hence measure OX on the u scale; it gives the number of ft.-lbs. represented by $w.s$.

(68) If s is actually 6″, which construction, (i) or (ii), would be most convenient?

(69) In the C.G.S. system the unit of work is an erg=dyne × centimetre. If d (Fig. 36) is the weight of a body in dynes and 1,000,000 dynes is represented by u, find the work done in lifting the body through the distance c.

*(70) The speed of a body is given by v (Fig. 37), where u represents a foot per second. Find the time the body takes to go a distance D, represented by d; f represents a foot.

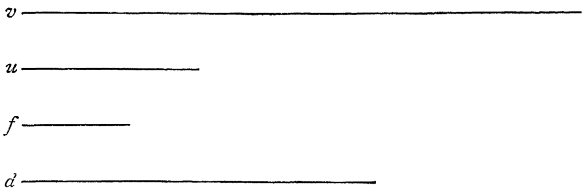

FIG. 37.

(71) The weight of a body is given by a line 4″ long, the lb. being represented by 1·3 cms. If 1 ft. is represented by a line of length 4″, find the work done in lifting the body through a distance given by a line of length 15″.

(72) Find graphically in ft.-tons the work done in raising a body, weight 0·75 ton, through a distance of 23·2 ft.

Moment of a Force. If a force be applied to the arm of a lever, the **turning moment** or **torque** of the force about the axis (fulcrum) of the lever is measured in magnitude by the product of the force and the perpendicular on its line of action from the axis. The geometrical representation of a moment is (like that of work done) an area. The difference between the two products we shall see later.

(73) A straight bar PQ, 12 ft. long, is hinged at Q, a force of 13 lbs. is applied at P making an angle of 35° with PQ. If a force of 1 lb. be represented by a line 1 cm. long and if 1 ft. be represented by 0·1″, find in two ways a line which represents the moment about Q, and read off the moment by scale.

MISCELLANEOUS EXAMPLES. I.

1. Draw the lines a, b and c of lengths 4·7, 3·9, and 5·2 cms. Find lines representing $\frac{A}{B}$, $A \cdot B$, $A\frac{B}{C}$, to the scale of 0·5″ to unity, and the numerical values of those quantities.

2. Find a line which represents the fraction $\frac{1}{7}$ to the scale of 9 cms. to unity.

3. Determine graphically the value of $\sqrt{6}$ and $\sqrt[4]{6}$.

4. Find a line whose length represents $\sqrt{7 \cdot 2}$ to the scale of 0·7 inch to unity, and read off the value of the square root.

5. Construct the line whose equation is $1 \cdot 7y = 5 \cdot 8x$, and from the line read off the values of $\frac{2 \cdot 3 \times 5 \cdot 8}{1 \cdot 7}$, $\frac{3 \cdot 7 \times 5 \cdot 8}{1 \cdot 7}$ and $\frac{4 \cdot 6 \times 5 \cdot 8}{1 \cdot 7}$.

6. Construct geometrically the curve $y = 2 \cdot 7x^2$ and find the values of $2 \cdot 7 \times 4 \cdot 1^2$, $3 \cdot 6^2 \times 2 \cdot 7$ and $\sqrt{\frac{7 \cdot 9}{2 \cdot 7}}$.

7. If a line of length 7·2 cms. represents unity, find the product of the ratio $\frac{A}{B}$ and C, where $a = 3 \cdot 48''$, $b = 1 \cdot 85''$ and $c = 1 \cdot 62''$, $2 \cdot 08''$, $3 \cdot 55''$, $4 \cdot 28''$ in turn.

8. In constructing the curve of cubes, unity is represented along Ox by $2''$ and along Oy by $0 \cdot 5''$. What is the relation between the *lengths* x and y for any point on the curve ?

9. Find the product $A \,.\, B$ in three ways, where $a = 8 \cdot 7$ cms., $b = 4 \cdot 8$ cms. and $u = 1 \cdot 62''$.

10. $w = 2 \cdot 3''$ represents, to the scale 2 cms. to 1 lb., the weight of a body ; $h = 7 \cdot 2$ cms. represents, to the scale 1 cm. to 1′, the vertical distance the body is moved through ; find the work done in ft.-lbs.

11. Construct geometrically the curve $xy = 3 \cdot 2$, and find the values of 3·2 times the reciprocals of 1·3, 2·7, 4·2 and 0·8.

12. By aid of a straight line divide 2·72, 0·85, 3·64, 1·88 in turn by 1·35.

*13. The volume of a pyramid being $\frac{1}{3}$ base area × height, find graphically the volume in cubic feet when the base is a rectangle, the sides of the rectangle being given by lines of 7·2 and 3·9 cms. and the height by a line of 4·3 cms., the scale being $2''$ to 1 foot.

14. To divide a set of numbers by 5·44 use a straight line graph, taking the vertical scale (for the numbers to be divided) as 1 quarter-inch for 10, and the horizontal scale (for the quotients) as 1 quarter-inch for 1. Obtain from your graph the quotients of 60 and 218 by 5·44, and verify by calculation with the tables. Explain why the graphical method gives the result. (Military Entrance Examination, 1905.)

15. Find graphically the values of 2·38, 18·3, 47·5 when multiplied by $\frac{0 \cdot 763}{5 \cdot 47}$.

CHAPTER II.

GRAPHICAL MENSURATION.

THE chief problem studied in this chapter may be concisely stated as follows: *Given an area bounded by straight or curved lines, to find a length which will represent to a given scale the magnitude of the area.*

The process for effecting this is called **reducing the given area to unit base.** The required length is the altitude of a rectangle whose base is the unit of length and whose area is equal to the given area.

The Triangle. The area of a triangle being half the product of the base and altitude, if we can find another triangle of equal area having one side twice the unit of length the altitude of this second triangle measures the area.

Method I. Transfer ABC (Fig. 38) to drawing paper. Draw through A a line parallel to BC; with B as centre, describe an arc of a circle of radius 2 inches cutting AA_1 at A_1 (or put a scale at B in such a position that $BA_1 = 2''$). By the aid of set squares, draw CD perpendicular to BA_1. Put the inch scale along CD and read off p, the number of square inches in ABC.

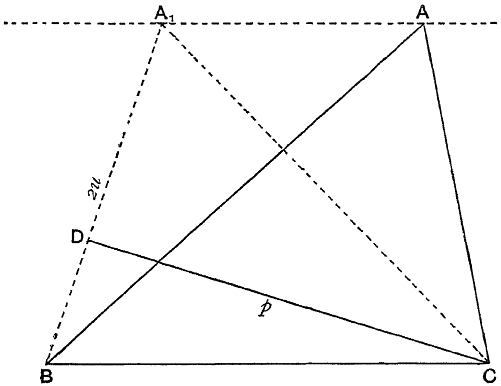

FIG. 38.

Proof. Since AA_1 is parallel to BC, the area of $ABC =$ area of A_1BC; and, since $BA_1 = 2$ inches,

$$\text{the area of } ABC = \tfrac{1}{2} \times 2 \times p = p \text{ (sq. ins.).}$$

(1) Draw a triangle having sides 3·7, 2·8 and 4·3 inches, and find a line giving its area in sq. ins. (In this case 2 inches is less than any perpendicular from a vertex to the opposite side, so take 4 inches for BA_1 and read p on the $\frac{1}{2}''$ scale.)

(2) Draw a triangle having sides 9, 7·3, 5·6 cms. and find a line giving the area in sq. cms. (Take BA_1, 10 cms. and read p in mms.)

Method II. Since the lengths of BA_1 and CD may be interchanged without altering the area (*i.e.* we may make $CD = 2$ inches or in general $= 2u$), if BDC be kept a right angle, the line from A to the base, parallel to BD, measures the area.

Transfer ABC (Fig. 39) to drawing paper. With C as centre describe an arc of a circle of radius 2 units. Place the set squares, in contact along one edge, so that an edge of one going through B is perpendicular to an edge of the other going through C; a position can easily be found for the set squares in which these edges intersect on the arc at D (say). In this position BD is the tangent to the arc at D and CD is the radius to the point of contact.* Move the B set square, parallel to itself, until the edge passes through A, and draw AE to cut the base in E. Measure AE $(=p)$; it gives the area of ABC (3·23 sq. ins.).

Proof. From (Fig. 39) we see that the areas ABC and A_1BC are equal, and that the area of the latter is

$$\tfrac{1}{2}A_1C \times \text{altitude} = \tfrac{1}{2}AE \, . \, 2u,$$

and hence AE measures the area in sq. units.

When does this construction fail? See that the difficulty can be got over by taking u, $\frac{1}{2}u$, $\frac{1}{4}u$, … instead of $2u$.

(3) Repeat this measurement by describing a semicircle on BC and setting off a chord, $CD = 2u$, in it, and then proceed as before.

(4) Find the area in sq. inches of the triangle whose sides are 7·5, 6·3 and 4·7 cms.

Method III. Transfer ABC (Fig. 40) to drawing paper. Set off along BC, $BD = 2u$. Mark the point E on AB where CE,

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* D could be found with the right angle of one set square only if the corner were perfect and not rounded by use.

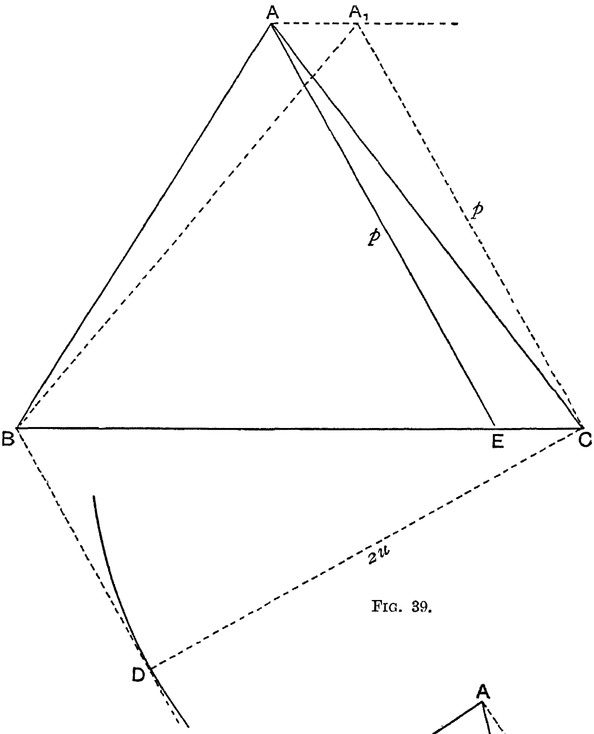
A
A₁
p
p
B
E
C
2u
D

FIG. 39.

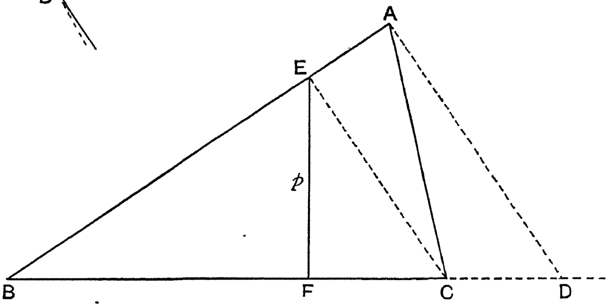

FIG. 40.

parallel to AD, cuts it; measure EF ($=p$), where EF is perpendicular to BC; p gives the area (1·58 sq. ins.).

Proof. Join DE;

then (as areas) $AEC = DEC$, $\therefore$ $ABC = EBC + DEC = EBD$, and the last triangle has $2u$ for its base.

(5) Repeat the construction, taking BA and AC as bases. Is this construction always possible?

Rectangle. Parallelograms and rectangles can be treated by the method given for quadrilaterals in the next section; but the following way is a little simpler.

Draw a rectangle $ABCD$ whose height BA is 8·5 cms. and base BC is 3·2 cm. To find its area in sq. inches, set off along BA, $BU = 1''$ (Fig. 41) and draw AE parallel to UC. Measure BE in inches, and the number so obtained is the area of $ABCD$ in sq. inches.

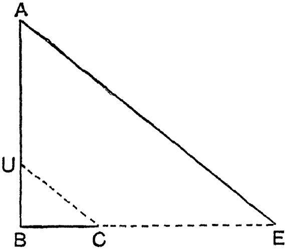

FIG. 41.

This is like the old construction of pp. 8 and 9 over again and needs no further demonstration.

Quadrilateral. Transfer the quadrilateral $ABCD$ (Fig. 42) to drawing paper. With B as centre, describe an arc of radius $2u$ ($2''$), and draw the tangent DE to it from D (or describe a semicircle on BD, and set off $BE = 2u$ in it). From A and C draw AA_1 and CC_1 parallel to BD, and measure A_1C_1 which gives the area of $ABCD$ (2·57 sq. inches).

Proof. Join BA_1 and BC_1; then (as areas) $ABD = A_1BD$ and $BDC = BDC_1$, $\therefore$ $ABCD = A_1BC_1$, a triangle whose altitude is $2u$ and base A_1C_1.

(6) The sides of a quadrilateral, taken in the order $ABCDA$, are 3·7, 2, 4 and 2·8 inches, the angle ABC is a right angle; find the area (i) by using the diagonal AC, (ii) by using BD.

(7) From a point O in a field lengths are measured $OA=35$ ft., $OB=72$ ft. and $OC=51$ ft., the angles AOB and BOC being 55° and 50° respectively. Draw the figure $OABC$ to scale (2 cms. to 10 ft. say). Reduce the figure to unit base, and determine the area marked out on the field by the contour $OABC$.

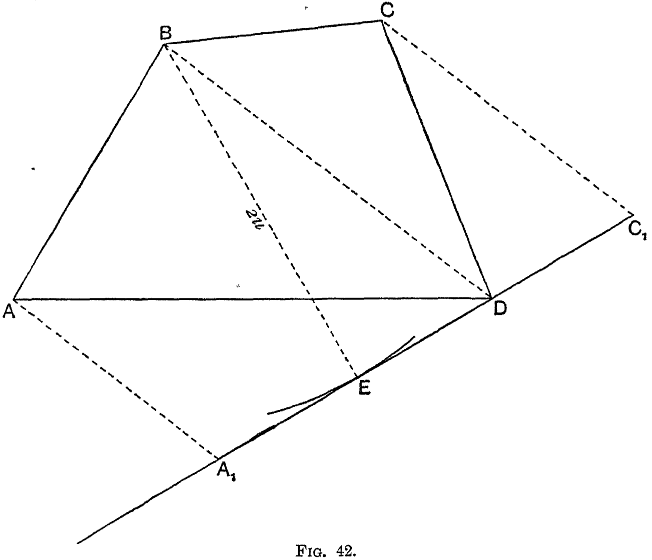

FIG. 42.

Re-entrant Quadrilateral. The construction already given holds for a re-entrant quadrilateral. Transfer $ABCD$ (Fig. 43) to drawing paper and proceed exactly as before. In this case, using the diagonal BD, A_1 and C_1 are on the same side of E.

Measure A_1C_1 in inches, this gives the area in sq. inches (0·97). Notice that A_1C_1 is now the difference between A_1D and C_1D instead of the sum

Proof. Join BA_1 and CA_1; $ABCD$ is now the *difference* between the triangles ABD and CBD, and $BAD=BA_1D$, $BCD=BC_1D$. Hence

$$BADC=BA_1D-BC_1D=BA_1C_1,$$

a triangle of altitude BE $(2u)$ and base A_1C_1.

*If C, in Fig. 42, be moved nearer and nearer to BD, BCD gets smaller and smaller and vanishes when C is on BD. If C be moved still further, so that it crosses BD, the triangle becomes negative and has to be subtracted from ABD. Corresponding to this change of sign of the area, there is a change in the sense of the boundary as determined by the order of the letters. In Fig. 42 the boundary, in the order of the letters BCD, is described clockwise, whereas, in Fig. 43, the boundary, taken in the same order, is described contraclockwise. On changing the sense of the boundary of an area, we must, therefore, change the sign of the area The equation $ABCD = ABD + BCD$ holds, therefore, for both Figs. 42 and 43, and since $BCD = -CBD$, we have

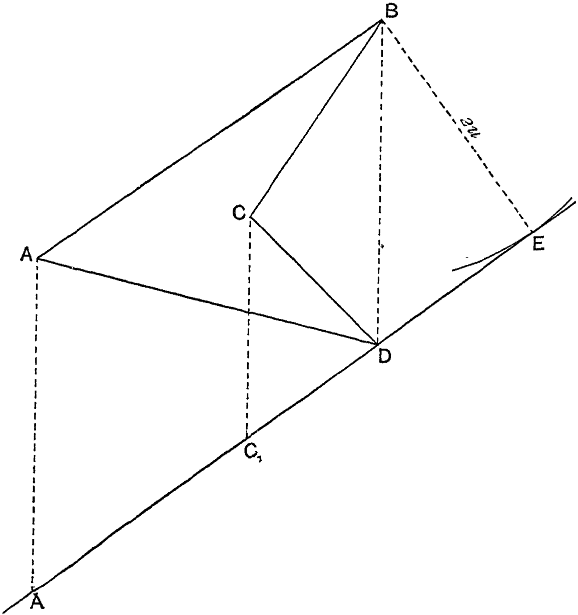

FIG. 43.

$$ABCD = ABD - CBD,$$

where the three areas have the same sense to their boundaries.

It is usual to consider an area, whose boundary is described contraclockwise, as positive, one with a clockwise boundary as negative. In Fig. 43 *ABCD* is a negative area, but *BADC* is a positive one.

(8) Given $BAD=60°$, $AB=7{\cdot}2$, $AD=6$, $BC=5$ and $CD=3{\cdot}3$ cms., find the area in sq. inches.

* **Cross Quadrilateral.** If *C* is taken on the other side of *AB* or *AD* the figure is called a **cross quadrilateral**, and the area is still the difference between *ABD* and *CBD*.

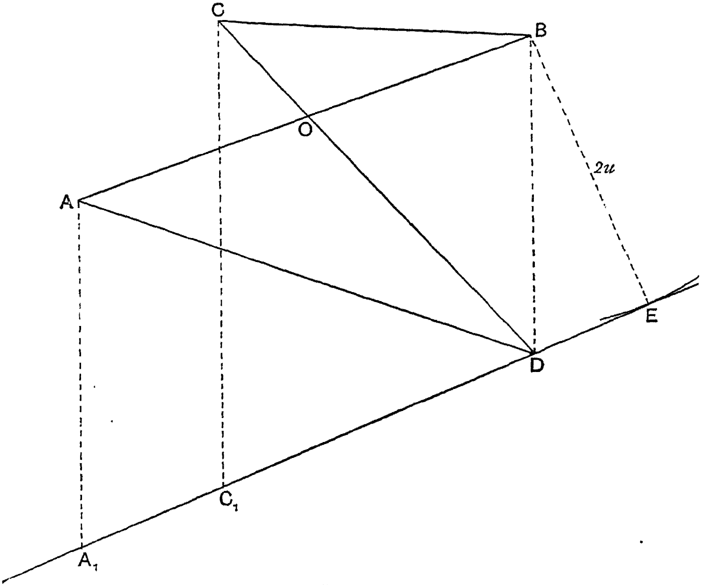

FIG. 44.

Transfer the annexed figure *ABCD* (Fig. 44) to paper and reduce to unit base as before. See that A_1C_1 (0·65″) still represents the area *ABD* – the area *DCB*. Hence the area *ABCD* is that of the triangle *AOD* – the triangle *BOC*. On going round the figure *ABCD* in the order of the letters from *A* back to *A*, it is seen that *AOD* is described clockwise and *BCO* contraclockwise.

The triangle AOD being greater than BOC, the figure $ABCD$, taken in the order of the letters, is negative, but $ADCB$ is positive.

(9) Given $AO=1{\cdot}2$, $DO=2{\cdot}1$, $CO=0{\cdot}4$ and $BO=0{\cdot}8$ inches, and $COB=110°$; find the area of the cross quadrilateral in sq. ins.

Polygons (including the quadrilateral as a particular case).

Transfer $ABCDEF$ (Fig. 45) to drawing paper. Produce AB both ways. With set squares mark C_1 on AB so that CC_1 is parallel to DB; mark D_1 so that DD_1 is parallel to C_1E, then on the other side mark F_1 so that F_1F is parallel to AE; find the area of EF_1D_1 by reducing to unit base; this is the area of $ABCDEFA$ (19 sq. cms.).

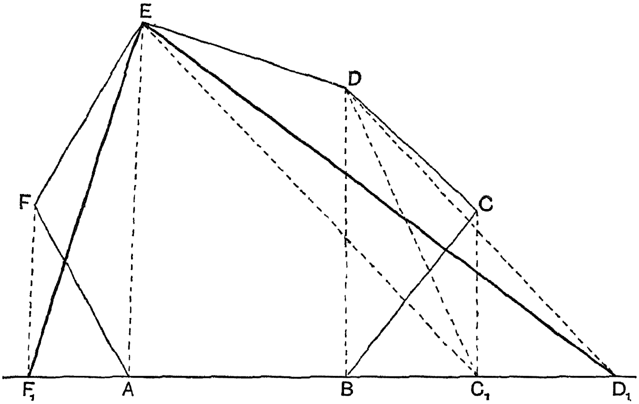

FIG. 45.

In general, to reduce any polygon to a triangle having its base on a side AB of the polygon, put a set square along the line joining B to the *next but one vertex* D, and bring down the omitted vertex C to C_1 on AB by a parallel to BD.

Then put the set square along the line joining C_1 to the *next but one vertex* E, and bring down D to D_1 on AB as before. This process is to be continued until only one vertex is left above AB. Should the construction lines become awkward, the transformation can be transferred to the end A of AB, or a new base line may be taken on another side of the polygon.

Proof. Since $BCD = BC_1D$ in area

the hexagon $ABCDEFA$ = the pentagon AC_1DEFA.

Join D_1E; then, since $C_1DE = C_1D_1E$

the pentagon AC_1DEFA = the quadrilateral AD_1EFA.

Join F_1E; then, since $AEF = AEF_1$

the quadrilateral AD_1EFA = the triangle F_1D_1E.

(10) Reduce the same polygon to a triangle having its base (i) on BC, (ii) on AD.

(11) Reduce the re-entrant pentagon $AEDCB$ (Fig. 46) to unit base.

$AB = 4{\cdot}5''$, $BC = 2{\cdot}88''$, $AE = 6{\cdot}06''$, $ED = 3{\cdot}8''$.

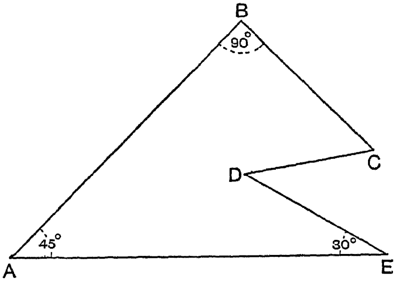

FIG. 46.

EXAMPLE. *To find the area between a polygon and one enclosed within it.*

Instead of reducing the two polygons separately, the mensuration may be effected by a continuous process.

Transfer the figure $ABC\ldots$ (Fig. 47) to drawing paper. Produce KH to cut FA at L, then $ABCDEFA - HKJIH$ = the re-entrant polygon $ABCDEFGHIJKLA$ when G and L are coincident.

Reduce this polygon to unit base as before (32·3 sq. cms.).

For many-sided figures the process is tedious and errors may easily accumulate unless very great care is taken. In such cases, and for curved boundaries, the strip division method (p. 58) may be used. The **planimeter**, where one is obtainable, is, however, best for such cases.

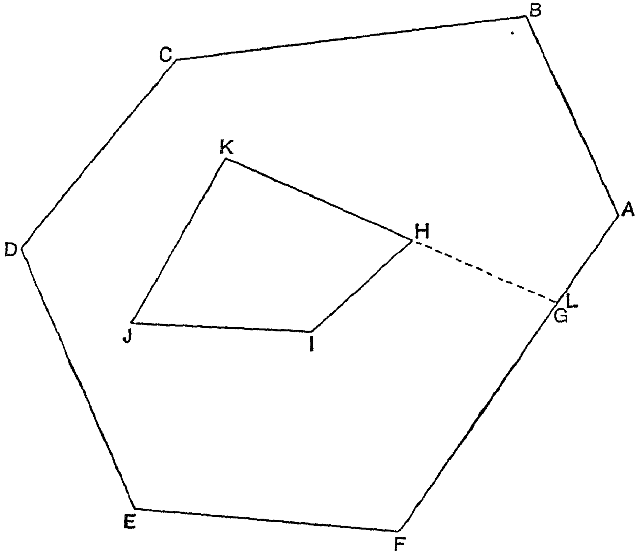

FIG. 47.

(12) *ABCDEA* is a small pentagonal field, the sides *AB*, *BC* and *AE* were measured and found to be 136, 52 and 95 yds. long respectively, and the angles *ABC*, *BCD*, *DEA* and *EAB* had magnitudes 75°, 70°, 60° and 50°. Draw a plan of the field to a scale of 1 cm. to 10 yds., reduce the area to unit base and determine the area of the field in sq. yds.

Circular Arc. Draw a circular arc *AB* (Fig. 48) of radius 3 inches subtending 90° at the centre. Draw the tangent AB_1 to this at *A*. Set the dividers so that the distance apart of the end points is about 1·5″. Step off the arc from *B* towards *A* with alternate clock- and contraclockwise sweeps, prick a point on AB_1 where the last semicircular sweep would cut it. From this point make as many steps along AB_1 as were taken for the arc, mark B_1 the end of the last step. Then AB_1 is very roughly the length of AB.

Adjust the dividers again so that the length of step is about $\frac{1}{2}''$ (roughly), and repeat the operation; you will come to a point B_2 near B_1.

Which most nearly gives the length of the arc BA, BA_1 or BA_2, and why?

Adjust the dividers again so that length of step is about $\frac{1}{4}''$ (roughly). Repeat the operation again, and find a point B_3 on AB. Can you distinguish between B_2 and B_3?

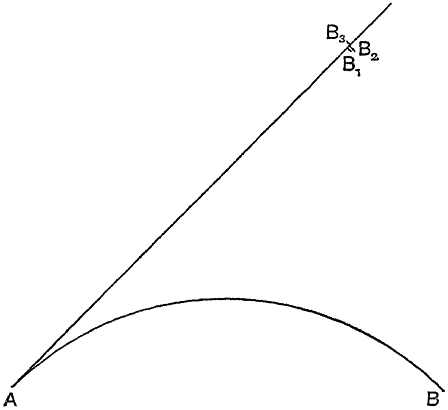

FIG. 48.

AB_3 is approximately the length of the arc AB.

Professor Rankine gave the following construction for finding as a straight line the length of a circular arc AB.

Produce the chord AB (Fig. 49) to C making $BC = \frac{1}{2}AB$. Draw the tangent at B, and with C as centre describe the arc AD cutting BD at D, then BD is the length of the arc AB approximately.

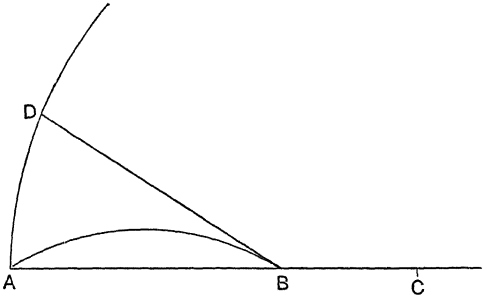

FIG. 49.

For an angle of 90° the error is about 1 %, so that the method should not be used for arcs greater than a quadrant.

(13) Draw a circular arc of radius 4″ and subtending 50° at the centre. Measure the length of the arc in cms. by the two methods given.

(14) Draw the circular arc whose base is 10·5 cms. and arc length 11·4 cms.

In this problem the length *AB* is given and therefore *AC* is known. The point *D* is therefore given by the intersection of two circles. The centre of the circular arc *AB* is therefore found as the point of intersection of the perpendicular at *B* to *BD*, and the perpendicular to *AB* at its mid-point.

For the arc of a semicircle there is a method due to a Polish Jesuit, Kochansky (*circa* 1685).

Draw a semicircle *ABC* (Fig. 50) of radius 3″. Draw a tangent at one end of the diameter (*B*); set off *BD* so that $BOD = 30°$. From *D* step off $DE =$ three times the radius, then *AE* is the length of the arc *ACB* nearly.[1]

Any arc greater than a semicircle can be now found by first finding the length of the semicircle by the above construction and the remainder by Professor Rankine's construction.

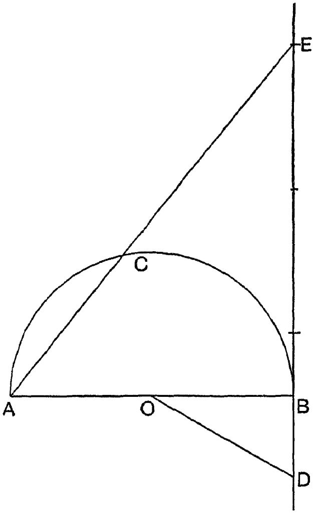

FIG. 50.

(15) Find the length of a semicircular arc of 4·72″ radius and compare with twice the length of the corresponding quadrant given by Rankine's rule and the length given by stepping off the arc along a tangent.

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[1] The theory of the construction gives a value of $\pi = 3·14153$ instead of 3·14159.

Area of Circular Sector. Draw a circular arc AB (Fig. 51) of radius 3·5″ subtending at the centre an angle of 70°. At any point C of the arc draw a tangent, and step off CB_1 equal to the arc CB, and CA_1 equal to the arc CA.

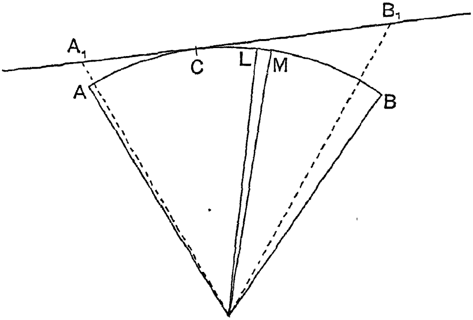

FIG. 51.

Reduce the triangle A_1OB_1 to unit base by one of the given methods, and measure the altitude in inches. This number is the area of the sector AOB in sq. inches.

Proof. If the arc be supposed divided up into a very great number of small parts LM (LM as drawn is not very small, but this is only because if it were very small the points L and M would seem to the eye coincident). LM being very small, it is very nearly straight, and its area is therefore approximately $\frac{1}{2}LM \times$ the perpendicular (p) from O on it.

In addition, if all the chords like LM are equal, the perpendiculars are all equal, and therefore

Area of sector AOB is approximately equal to

$$\tfrac{1}{2}p \times (\text{the sum of the steps } LM)$$
$$= \tfrac{1}{2}p\Sigma LM. \quad \text{(Read: "the sum of all terms like } LM\text{.")}$$

As the number of steps increases indefinitely the length LM diminishes without limit; but ΣLM approaches and ultimately becomes the arc AB, p becomes the radius r of the arc, while $\frac{1}{2}p\Sigma LM$ becomes the area of the sector.

Hence, area of the sector = area of A_1OB_1.

The formula for the area is: Area $=\frac{1}{2}r^2\theta$,

where r is the radius, θ the circular measure[1] of the angle, and $r\theta$ the length of the arc.

---

[1] See Note A, p. 374.

(16) Draw a circular sector of radius 8 cms. such that the base of the segment is 9 cms. Find the area in sq. inches.

(17) Find the area of a quadrant of a circle of radius 3·4 inches.

Area of a Segment of a Circle.

(i) **Segment less than a semicircle.** Draw a sector $AOBC$ (Fig. 52) of angle 75° and radius 4″; join AB, cutting off the segment ABC. Set off the arc along the tangent at C, $CB_1 = CB$ and $CA_1 = CA$.

Join BB_1 and draw OB_2 parallel to it; similarly, draw OA_2 parallel to AA_1, then $A_1A_2B_2B_1$ has the same area as ABC.

Reduce this quadrilateral to unit base, and measure the area in sq. inches.

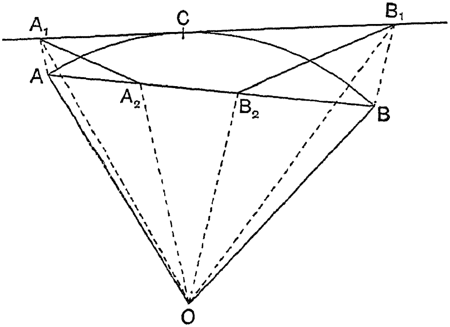

FIG. 52.

* **Proof.** Since segment ACB

$$= \text{sector } OACB - \triangle OAB = \triangle OA_1B_1 - \triangle OAB$$

$$= \triangle OA_2B_2 + \triangle OA_1A_2 + \triangle OB_2B_1 + A_2A_1B_1B_2 - \triangle OAB$$

$$= \triangle OA_2B_2 + \triangle OAA_2 + \triangle OB_2B - \triangle OAB + A_2A_1B_1B_2$$

$\therefore$ segment $= A_2A_1B_1B_2$.

A simplification is effected by taking C at A, so that A_2 is at A and the quadrilateral reduces to a triangle AB_1B_2. Draw the figure and make the construction. The proof then simplifies to:

$$\text{segment } ABC = \text{sector } AOBC - \triangle OBA$$

$$= \triangle AOB_1 - \triangle AOB_2 - \triangle OBB_2$$

$$= \triangle AOB_1 - \triangle AOB_2 - \triangle OB_1B_2$$

$$= \triangle AB_2B_1.$$

Find the area of this triangle and compare it with the previous result.

(18) Take $AOB=60°$, $OA=3''$. Find the area graphically, and compare with the value of $4{\cdot}5\left(\frac{\pi}{3}-\frac{\sqrt{3}}{2}\right)$.

(The formula giving the area is $A=\frac{r^2}{2}(\alpha-\sin 2\alpha)$, where r is the radius and $AOB=2\alpha$ in circular measure.)

(ii) **Segment greater than a semicircle.** Draw a segment ACB (Fig. 53) of a circle such that $AOB=150°$, and repeat the previous construction (p. 56). A crossed quadrilateral $A_1A_2B_2B_1$ is obtained. Reduce this to unit base.

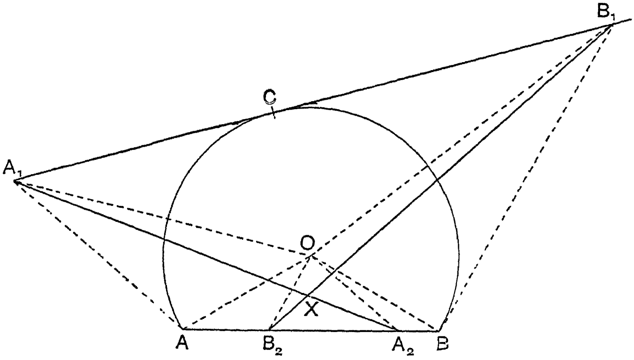

FIG. 53.

Verify, by means of the formula in Ex. 18.

* **Proof.** This is very similar to the preceding one.

$$\begin{aligned}\text{Segment } ACB &= \text{sector } OACB + \triangle AOB\\ &= A_2A_1B_1B_2 + AO_2B_2 - OA_2A_1 - OB_1B_2 + AOB\\ &= A_2A_1B_1B_2 + OA_2B_2 - OA_2A - OBB_2 + AOB\\ &= A_2A_1B_1B_2.\end{aligned}$$

For the same segment, draw the tangent at A and step off the arc along it. The area is thus reduced to a triangle AB_2B_1. Find the area of this triangle and compare it with the previous result.

(19) Reduce a semicircle of radius 7·3 cms. to unit base.

(20) Reduce a segment having three quadrants to its arc to unit base, the radius being $3{\cdot}2''$.

Irregular and Curved Figures. The areas of figures bounded by curves or many straight lines are best obtained by a planimeter. The method of strip division gives, with care, a very good approximation.

Place a sheet of tracing paper over the accompanying figure and draw its outline. Divide the distance between the two

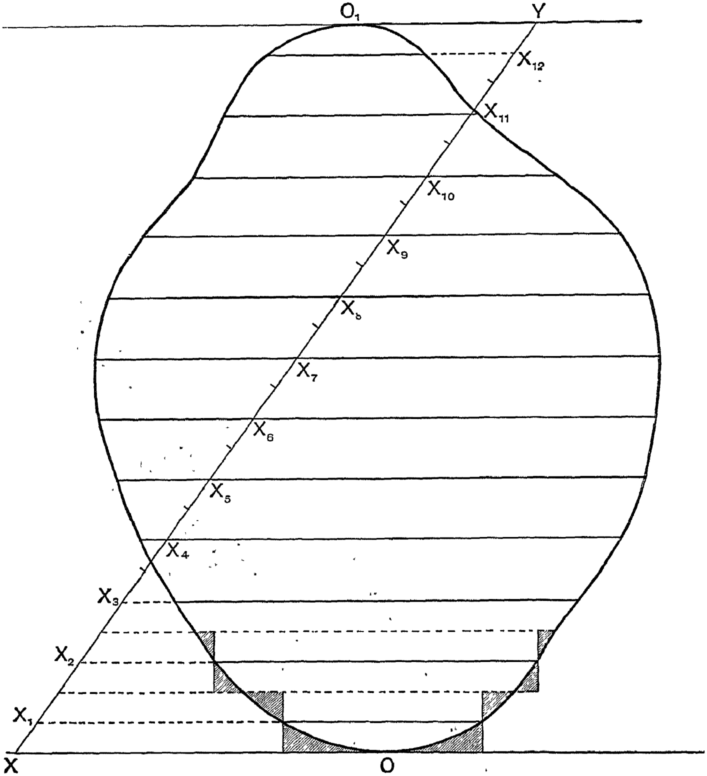

FIG. 54.

extreme points O and O_1 into 12 strips of equal width. To do this, set a scale slantwise between two parallel lines drawn at O and O_1 (the whole figure must lie between these parallels), in such a position XY that there are 12 equal divisions between X and Y.

Mark the mid-points X_1, X_2, ... of these divisions, and through these points draw lines parallel to XO.

Take a long strip of paper with a straight edge and add up graphically the segments of the mid-lines intercepted by the curve.

The given area is approximately that of a rectangle whose base is the width of the strips and whose altitude is the length on the long strip. Calculate this in sq. cms. (53·2).

The approximation made is in the assumption that each strip has the area of a rectangle whose height is the line through the mid-point (the mid-ordinate) and of the given width, and this, again, assumes that the little shaded areas outside the figure are just balanced by the shaded areas inside.

A more accurate value of the area is obtained by using **Simpson's Rule**, viz.

$$\text{Area} = \frac{h}{3}[y_0 + y_n + 4(y_1 + y_3 + \dots) + 2(y_2 + y_4 + \dots)],$$

where the area is divided into an **even** number of equally wide strips of width h; y_0 is the length of the first, and y_1 the length of the second side of the first strip; similarly, y_1 is the length of the first, and y_2 the length of the second side of the second strip, and so on; shortly put, y_0, y_1, y_2, ... y_n are the lengths of the bounding straight lines of the strips. Since the number of strips is even, n must be even. The rule only holds for an even number of strips.

Apply the rule to determine the area of Fig. 54. Here, taking the strips from bottom to top, $y_0 = 0$, y_1 is the length of the first dotted line within the curve, y_2 the second; the other y's are not shewn, but $y_{12} = 0$ ($n = 12$). Add the ordinates according to the rule, and compare the result with that obtained by the mid-ordinate rule.

Place the tracing paper over a sheet of squared mm. paper and count up the number of large squares wholly within the curve, and then the number of small ones between these squares and the curves, estimating for any decimals of a small square. Find the area by this means and compare with the previous results.

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(21) Obtain the area of the hexagon on p. 50 by the strip division method.

(22) Draw a semicircle of radius $3{\cdot}1''$ on squared paper.

Find the area by

(1) The mid-ordinate rule. (2) Simpson's Rule.
(3) Counting the squares. (4) Calculating from formula : Area $=\frac{1}{2}\pi r^2$.

* Another Method for Figures bounded by Curved Lines.

For these figures it is usual to assume that the boundary can be divided into parabolic arcs. In most elementary text books on Geometrical Conics, it is shewn that the area of any parabolic segment, such as ACB (Fig. 55), is equal to $\frac{4}{3}$ of the triangle having as base the base of the segment, and having its vertex on the tangent to the curve parallel to the base. In Fig. 55 $\frac{4}{3}$ of triangle $ACB =$ area of segment.

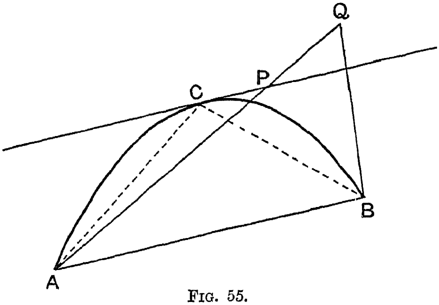

FIG. 55.

We may, therefore, choose any convenient point P on the tangent, divide AP into three equal parts, produce to Q, where $PQ =$ one of these parts, and join QB. Then the area of AQB is the parabolic segment area

(23) Find the area of the circular segment of Exercise (16) by treating it as a parabolic segment.

Any curved area such as $ABCDEA$ (Fig. 56) may be divided into a number of approximately parabolic arcs, AB, BC, ... Each arc must be curved in one way only (*i.e.* must not contain a point of inflexion) and through considerably less than 180° : the procedure is as follows. Start with the base BC say; draw a tangent parallel to BC and produce AB to cut it at K; divide BK into three equal parts and make BB_1 equal to four of these; join CB_1. Then $CBB_1 =$ given parabolic segment in area. Again, produce B_1C to cut the tangent parallel to CD, and take the $\frac{4}{3}$ point C_1 on B_1C and join C_1D. Then $CC_1D =$ area of corre-

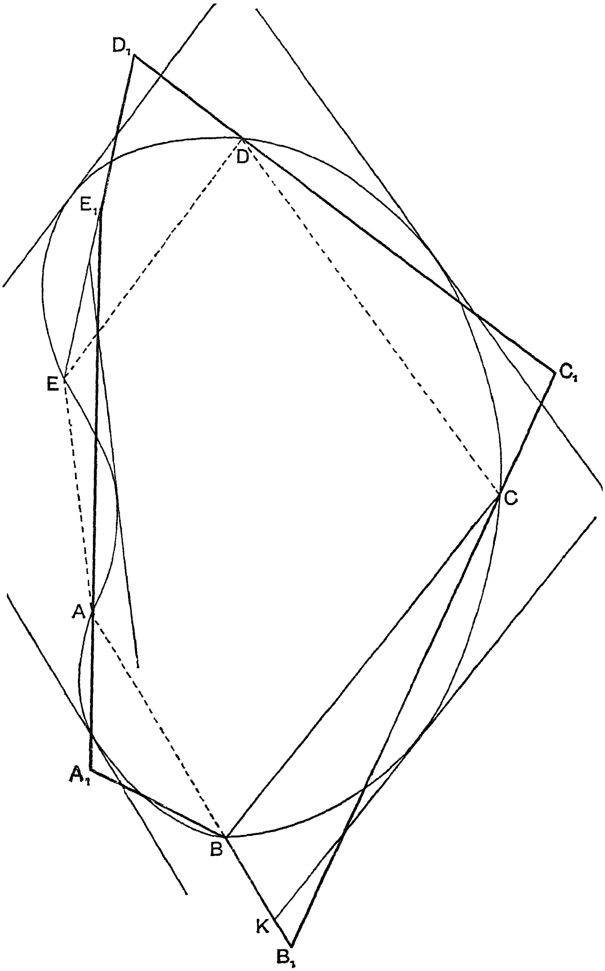

FIG. 56.

sponding segment. Proceed in this way round the curve and reduce the curved area to the rectilinear one $B_1C_1D_1E_1A_1B$. Reduce this to unit base in the usual way (6·9 sq. ins.).

This method is, however, rather tedious, and errors due to want of parallelism may, unless very great care be taken, lead to considerable final error. It is generally better to use the method of strip division and either the mid-ordinate or Simpson's Rule for such cases.

Volumes of Revolution. Any such volume may be found by a double reduction. Suppose the given area in Fig. 57 to revolve about the line XX; then it will generate a figure called a **volume of revolution.** In particular, if a right-angled triangle ABC revolve about its base AB, it will generate a cone; if a semicircle revolve about its diameter, it will generate a sphere; a rectangle about its base will generate a cylinder; any triangle about one side will generate a spear-head volume; a circle about an exterior line in its plane will generate an anchor ring; and a rectangle about a line parallel to its base will generate a figure like the rim of a fly-wheel.

The construction to be explained is one, therefore, of great generality.

Make a tracing of the outline of the given figure.

Divide the area up into 10 equally wide strips, parallel to XX, and draw the mid-lines for each of these as in Fig. 57. Only the mid-lines are shewn.

Draw a line YY above XX and parallel to it at a distance a inches, given by $a=\frac{10}{\pi}$ (that in Fig. 57 is purposely not at the correct distance).

Project the end points, like AB of each mid-line up to $A'B'$, on YY, AA' and BB' being perpendicular to YY. Join A' and B' to any fixed point O on XX, cutting AB in A_1 and B_1. Connect all the points like A_1B_1 by a curve. The area of this curve is proportional to the volume. If a planimeter is not obtainable, add up all the mid-lines like A_1B_1 of the new figure (called the **First Equivalent Figure**) by the straight-edged paper method, and obtain the area approximately in square inches; multiply this by 20; the result is the volume of revolution in cubic inches.

*__Proof.__ Suppose the area divided up, not into 10 only, but into a very great number of equally thin strips, of which AB may represent any one. The strip must be considered infinitely thin, so that it is all at the same distance from XX. Let x be the length AB, and h the width of the strip, then, however small h may be, Σhx (read: "the sum of all terms like hx") will be the area.

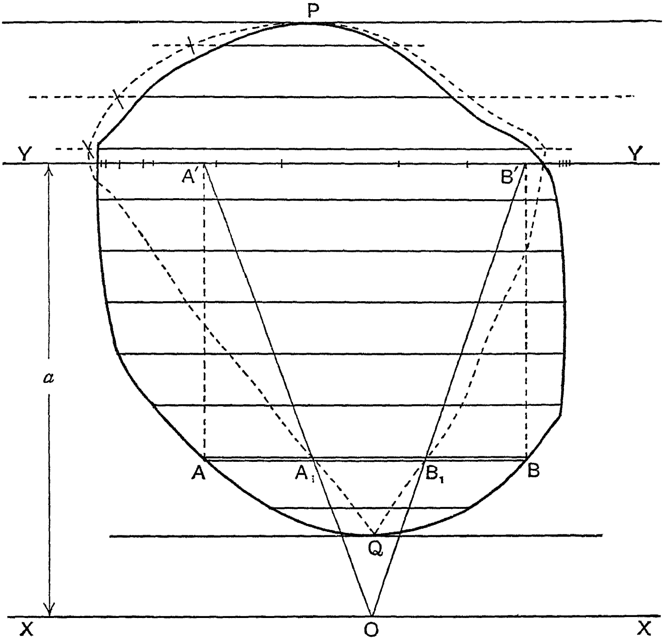

FIG. 57.

Let y be the distance of AB from XX, then when the strip revolves round XX, keeping always at the same distance y from it, it will generate a very thin hollow cylinder. The height of this cylinder is x, its circumference is $2\pi y$ and its thickness is h; its volume, therefore, will be $2\pi yxh$.

All the other strips into which the area has been divided, will, on revolution about XX, also generate very thin hollow cylinders, and the sum of all these hollow cylinders is the volume of revolution of the original area. Hence the

$$\text{volume of revolution} = \Sigma 2\pi yhx.$$

In this sum, x and y change from term to term, but $2\pi h$ is the same, and is therefore a common factor to all terms in the sum. Hence, if V is the volume,

$$V = 2\pi h\, \Sigma xy.$$

Let a be the distance of YY from XX, then, from the method of constructing the First Equivalent Figure, we see that OA_1B_1 and $OA'B'$ are similar triangles, and therefore

$$\frac{A_1B_1}{y} = \frac{A'B'}{a} = \frac{AB}{a} = \frac{x}{a},$$

hence $$xy = a \,.\, A_1B_1.$$

A similar equation holds for all the strips like AB, and hence

$$\begin{aligned} \Sigma hxy &= \Sigma ahA_1B_1, \\ &= a\, \Sigma hA_1B_1. \end{aligned}$$

But hA_1B_1 is the area of one strip of the Equivalent Figure, and ΣhA_1B_1 is therefore the whole area.

$$\therefore\ \Sigma hxy \text{ or } h\Sigma xy = a \times \text{area of Equivalent Figure},$$

$$\therefore\ 2\pi h\, \Sigma xy = 2\pi a \times \text{area},$$

$$\textit{i.e. } V = 2\pi a \times \text{area of Equivalent Figure}.$$

If, then, in our special case

$$a = \frac{10''}{\pi},\ \ 2\pi a = 20''$$

and $$V = 20 \times \text{area of Equivalent Figure},$$

and is in cubic inches if the area be in square inches.

(24) Find the volume generated by a right-angled triangle ABC in revolving about its base AB, if $AB = 4''$, $BC = 3''$.

*(25) Find the volume of a sphere of radius 1·73″.

*(26) The coordinates of three points A, B, C are, in inches, (0·5, 0·6), (1, 2·5), (3·5, 0). Find the volume generated by the revolution of ABC about the axis of x. Take point (3·5, 0) as O.

*(27) Draw a segment of a circle of base 4·4″ and height 2·9″. Find the volume generated by revolving the segment about its base.

MISCELLANEOUS EXAMPLES. II.

1. Reduce an equilateral triangle of side 9·1 cms. to unit base $u=1''$ by the three methods given, and compare these determinations of the area with those obtained by (i) measuring the base and altitude and taking half the product, (ii) by calculation from the formula.

$$\text{Area}=\frac{1}{2}\cdot a^2\cdot\frac{\sqrt{3}}{2},$$

a being the length of a side in inches. (1 inch = 2·54 cms.)

2. The coordinates of four points A, B, C, D are (2·3, 0), (4·1, 2·1), (1·2, 4·9), (−0·8, 2·2) in inches. Find the areas of the quadrilaterals $ABCD$ and $ACDB$.

3. Find the area of the figure $ABCDE$ (Fig. 58) in square inches, where $AB=5{\cdot}7$ cms., $BC=9{\cdot}8$ cms., and AED is a circular arc for which $EF=3$ cms.

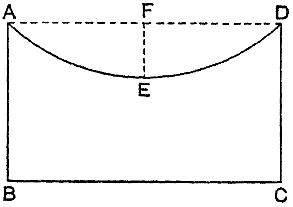

FIG. 58.

4. The coordinates of 5 points A, B, C, D, E are (1·1, 2·2), (4·9, 0·8), (7·3, 5·8), (5·1, 7·3), (0·8, 5·2) cms. Find the area of $ABCDE$ in sq. ins.

5. Find also the area of $ACDEB$.

[If O be the point of intersection of AC and BE, then the difference of the areas $OCDE$ and ABO has to be found; the construction for effecting the reduction to unit base is exactly as on p. 50.]

6. Reduce to unit base the area of the lens section shewn (Fig. 59), $AB=4''$, $CE=1{\cdot}8''$, $ED=1{\cdot}5''$, (i) by drawing the figure to scale on squared paper and counting up the contained squares; (ii) by reducing the segments separately to unit base.

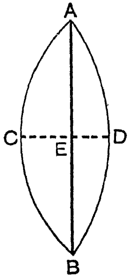

FIG. 59.

7. Draw a circular sector of radius 12 cms. and angle 150 degrees. Find the area of the *segment* (i) by the construction given in the text, (ii) by treating it as a parabolic segment, (iii) by dividing it up into eight strips of equal width and adding the mid-lines.

*8. The circular segment of question 7 revolves about its base; find the volume of the solid generated.

9. One side of a field is straight and of length 200 ft. At distances increasing by 20 ft. from one end, the width is measured and found to be 70, 100, 100, 130, 137, 180, 150, 145, 100 ft. Find approximately the area of the field.

10. Reduce to unit base the area $ABCD$, where $AB=5\cdot9$, $AD=8\cdot7$ cms. and AB is perpendicular to AD, and BCD is a circular arc of 12·5 cms. length, convexity outwards. (The arc may be drawn by reversing Rankine's construction.)

11. The corners of a triangular field PQR are determined with reference to a base line AB by the dimensions $PAB=57°$, $PBA=94°$, $QAB=64°$, $QBA=111°$, $RAB=130°$, $RBA=47°$. AB is 50 feet long. Draw a diagram to a scale of an inch to 100 feet, and determine the area of the field.

(Military Entrance Examination, 1905.)

12. A triangle has sides of 3·9, 3·2 and 4·2 inches. Draw the triangle, measure each of the angles with a protractor and find the area.

13. Test or prove geometrically the accuracy of the following graphical method of determining the area of a quadrilateral $ABCD$: "Join BD; through C draw CE parallel to BD meeting AB, produced if necessary, in E; with centre E and radius equal to twice the unit of length, describe a circle; from A draw a tangent to this circle, to meet DX, which is parallel to AB, in X. Then the number of units of length in AX is the number of units of area in $ABCD$." Data for the test figure: $BD=2$ in., $AB=1\cdot6$ in., $BC=1\cdot8$ in., $CD=1\cdot6$ in., $DA=1\cdot3$ in.

(Military Entrance Examination, 1905.)

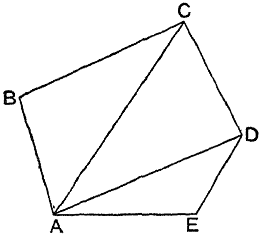

FIG. 60.

14. In a survey of a field $ABCDE$, of which a sketch is given (Fig. 60), the following measurements were made: $AB=84$ yards, $AC=173$ yards, $AD=175$ yards, $AE=130$ yards, $\angle BAC=42°$, $\angle CAD=36°$, $\angle DAE=20°$. Draw a plan to a scale of an inch to 30 yards, and find the area of the field from your plan.

(Naval and Engineer Cadets, 1904.)

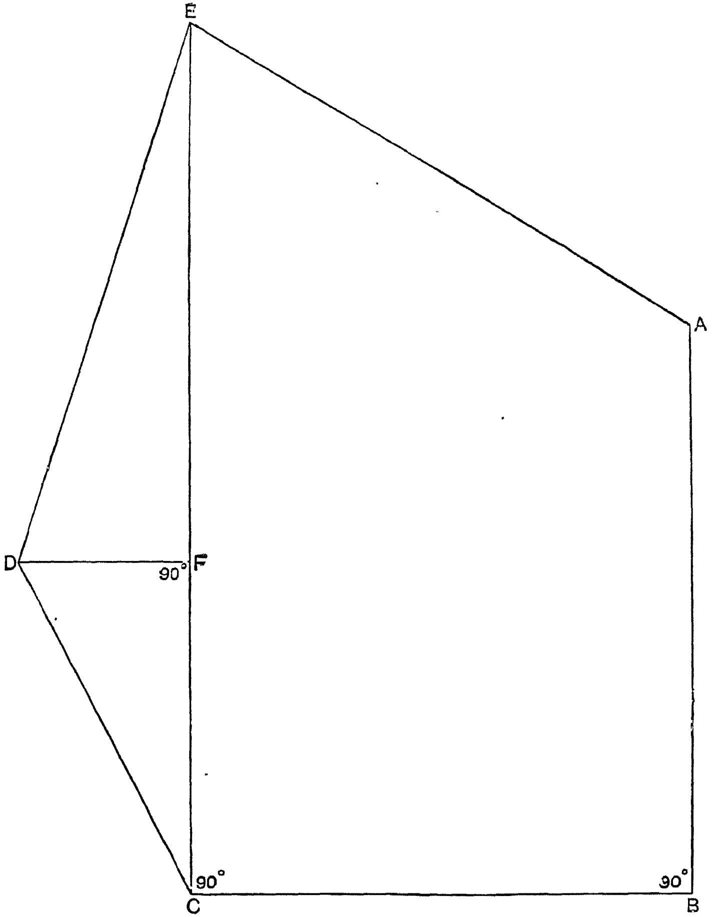

FIG. 61.

15. Above (Fig. 61) is a rough plan of the city of Paris drawn to the scale of 1 centimetre to the kilometre. Find the area of the city in square kilometres by measuring any lines you like. (Naval Cadets, 1903.)

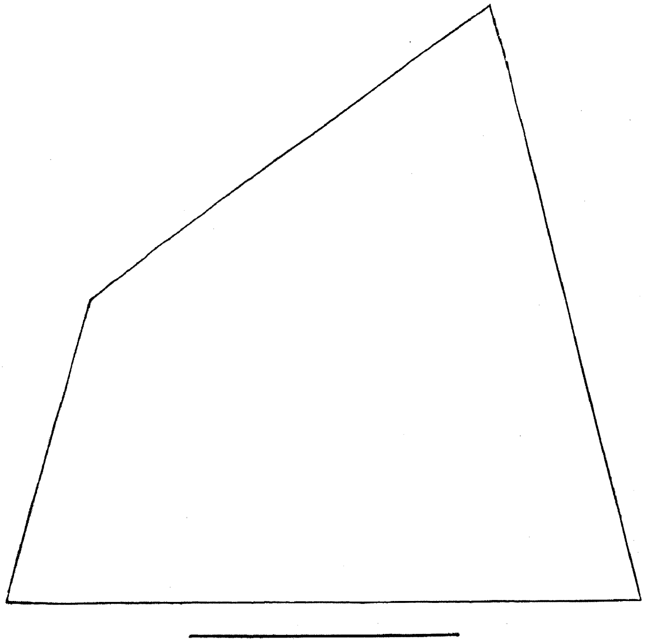

FIG. 62.

16. In any way you please, find the area of the given figure (Fig. 62) to the nearest square inch. State your method.

(Naval and Engineer Cadets, 1905.)

CHAPTER III.

VECTORS AND THEIR APPLICATION TO VELOCITIES, ACCELERATIONS, AND MASS-CENTRES.

Displacement. If a point moves from O to A (Fig. 63) along some curved or straight path, the line drawn *from* O *to* A is the **displacement of the point.** This displacement is independent of the actual path of the point, and depends only on the relative positions of the new and the initial points.

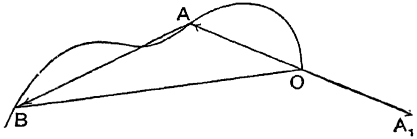

FIG. 63.

To specify the displacement, there must be given not only the **magnitude** of OA (*e.g.* 1·32 inches), but the **direction** or **lie** of the line (*e.g.* the North-South line); and not only the direction of the line, but the **sense** of the motion in that line (*e.g.* towards the North).

The displacement $\mathbf{OA}_1$ though equal to $\mathbf{OA}$ in magnitude and direction is of the opposite sense.

Displacements may be represented by lines drawn to scale, if the lines be placed in the proper directions and given the required senses. The sense of the displacement is indicated by an arrow head on the line.

To indicate that the line from O to A involves direction and sense as well as magnitude, it is convenient to print the letters in block type; thus $\mathbf{OA}$ means the length OA in its proper direction and with its correct sense.

In writing it is extremely difficult to keep the distinction between the block and the ordinary capital, and so, when writing, it is better to use a bar over the letters; thus, $\overline{OA}$ ("Maxwell" notation) means the same thing as **OA**.

Sum of Two Displacements. If the point moves from O to A and then to B, the final displacement from O is **OB**, while the displacement from A is **AB** and that of A from O is **OA**. These facts are symbolised by the equation

$$\mathbf{OB} = \mathbf{OA} + \mathbf{AB}.$$

Such an equation does not mean that the length OB is the sum of the lengths OA and AB; but simply, that the final position of the moving point is the same whether displaced directly from O to B, or first to A and then from A to B.

Sense and Sign. If the second displacement brings the point from A to O (so that B is at O), then the final displacement is zero, and

$$\mathbf{OA} + \mathbf{AO} = 0, \text{ or } \mathbf{OA} = -\mathbf{AO}.$$

Hence, changing the sense of a displacement changes the sign of its symbol. (See also p. 6.)

EXAMPLE. *A train travels due N. for* 20 *miles, then N.E. for* 10 *miles, what is its displacement? Another train goes* 10 *miles N.E. and then* 20 *miles N., show that its total displacement is the same as that of the first train.*

Represent 10 miles by 0·5 in.

(i) Set off $OA = 4''$ (Fig. 64) vertically upwards and $AB = 2''$ making 45° with OA produced. Measure OB in $\frac{1}{10}$ in. and divide mentally by 2; this gives the magnitude of the displacement (28). With a good protractor (the vernier protractor is best), or by aid of a scale of chords, measure θ (15°); then the displacement of the train is 28 miles 15° to the East of North (approximately). Measure θ also by the aid of p and a table of sines.

Notice that **OB** *itself* with the arrow head gives the displacement, but, if it is necessary to state the displacement in words, we must give not only its magnitude but also the direction and sense compared with some standard direction and sense—in this case the line drawn *towards* the North.

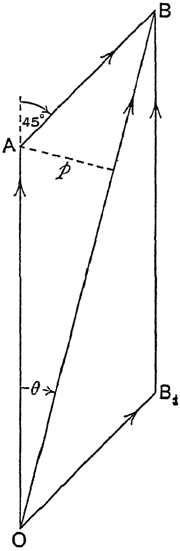

FIG. 64.

[It is perhaps as well to notice that, drawing *OA* parallel to the bound edge of the paper and towards the top of the page, is only the conventional way of representing "towards the North." The line *OA* will only represent the true displacement when the book is placed so that the arrow head on *OA* does point due North.]

(ii) Set off $OB_1 = 2''$ at 45° with the N. line, then 4″ due N.; arrive at B as before, for OB_1BA is a parallelogram.

Order of Addition. The order in which two displacements are added is immaterial; as an equation

$$\mathbf{OA} + \mathbf{AB} = \mathbf{OB}_1 + \mathbf{B}_1\mathbf{B},$$

and the displacement $\mathbf{B}_1\mathbf{B}$ is equal to **OA** and $\mathbf{OB}_1$ is equal to **AB**.

(1) A circus horse trots with uniform speed round a circus of radius 80 feet in 1 minute. Starting from the south position, give the displacement in 15, 30, 45 and 60 seconds. Make the drawing to the scale 2 mms. to a foot.

(2) A ring slides 5 ft. along a 7 ft. rod from E. to W. whilst the rod moves parallel to itself 7 ft. S.E. Find the total displacement of the ring. The rod now rotates through 70° about its West end in a clockwise sense, the ring remaining in the same position relatively to the rod; find the total displacement of the ring, if it starts at the E. end of the rod.

Addition of any number of Displacements.

EXAMPLE. *A point is displaced successively from* O *to* A, *from* A *to* B, *to* C *and to* D, *the displacements being given in magnitude, direction and sense by* $\mathbf{O_1A_1}$, $\mathbf{O_1B_1}$, $\mathbf{O_1C_1}$, $\mathbf{O_1D_1}$. *Find the resultant displacement.*

$$O_1A_1 = 4\cdot2, \quad O_1B_1 = 7\cdot92, \quad O_1C_1 = 10\cdot5, \quad O_1D_1 = 4\cdot1 \text{ cms.},$$
$$A_1O_1B_1 = 25^\circ, \quad B_1O_1C_1 = 120^\circ \text{ and } C_1O_1D_1 = 80^\circ.$$

From any point O draw OA (Fig. 65) equal and parallel to O_1A_1; from A draw AB equal and parallel to O_1B_1; from B draw BC equal and parallel to O_1C_1; and, finally, CD equal and parallel to O_1D_1. Then **OD** is the sum of the displacements

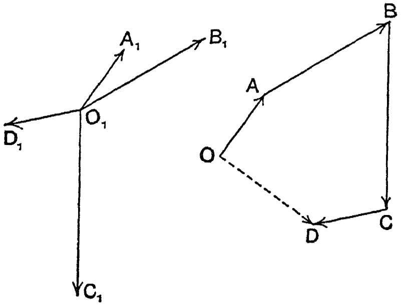

FIG. 65.

$$\mathbf{OD} = \mathbf{OA} + \mathbf{AB} + \mathbf{BC} + \mathbf{CD} = \mathbf{O_1A_1} + \mathbf{O_1B_1} + \mathbf{O_1C_1} + \mathbf{O_1D_1}.$$

Measure OD and the angle AOD; these measurements give the displacement in magnitude, direction and sense.

(3) From the same point O add the displacements in a different order, *e.g.* find

$$\mathbf{O_1A_1} + \mathbf{O_1C_1} + \mathbf{O_1D_1} + \mathbf{O_1B_1}$$
$$\mathbf{O_1D_1} + \mathbf{O_1B_1} + \mathbf{O_1A_1} + \mathbf{O_1C_1},$$

and shew that the same resultant displacement is obtained.

(4) Find the sum of the displacements

$$\mathbf{O_1A_1} - \mathbf{OB_1} + \mathbf{OC_1} - \mathbf{OD_1}$$

and

$$\mathbf{O_1C_1} - \mathbf{O_1B_1} - \mathbf{O_1D_1} + \mathbf{O_1A_1}.$$

Since $\mathbf{OA} = -\mathbf{AO}$, to subtract a displacement **AO** we have only to change the sense and add.

Relative Displacement. All displacements are relative, for there is no point in space known to be fixed. The earth turns on its axis, the axis moves round the sun, and the sun itself is moving in space.

EXAMPLE *Given the displacement of two points* A *and* B *relative to* O, *to find the displacement of* B *relative to* A.

In Fig. 66 **OA** and **OB** are the displacements relative to O; then **AB** is the displacement of B relative to A.

But

$$\mathbf{AB} = \mathbf{AO} + \mathbf{OB};$$

$$\therefore\ \mathbf{AB} = \mathbf{OB} - \mathbf{OA},$$

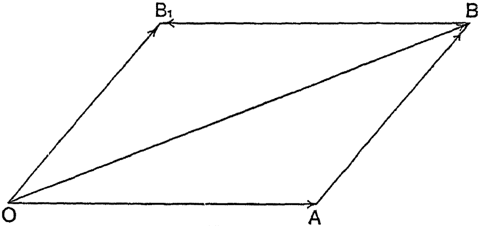

FIG. 66.

and **AB** is the difference of the displacements of A and B relative to O.

If the displacement of A relative to B, viz. **BA**, had been required we should have had

$$\mathbf{BA} = \mathbf{BO} + \mathbf{OA} = \mathbf{OA} - \mathbf{OB}.$$

Complete the parallelogram $OABB_1$ then

$$\mathbf{AB} = \mathbf{OB}_1 = \mathbf{OB} + \mathbf{BB}_1.$$

The displacement of B relative to A may therefore be regarded as follows. Give to both A and B a common displacement $\mathbf{BB}_1 = \mathbf{AO}$, making the total displacement of A zero; then the total displacement of B is the relative displacement required.

Vectors and Vector Quantities. Displacements are examples of **directed**, or **vector quantities.** The magnitude of the quantity—*e.g.* how many feet displaced—being represented to scale by a length, then, if this length be placed in the proper direction and given the proper sense, we have a complete representation of the vector quantity. Velocities of 20 and 30 miles an hour due N. and E. respectively are represented by lines of lengths 2 and 3 inches respectively, if the lines be placed in the given directions and arrow heads marked on them to give the senses: the scale is 1 inch to 10 miles an hour.

To specify a vector, magnitude, direction and sense must be given, and a **vector is defined as a geometrical quantity** (*e.g.* **a line) which has magnitude, direction and sense.**

Position. A vector has no definite position, it may be conceived as occupying any one of an infinite number of parallel positions.

In Fig. 67 **AB** and **CD** are equal vectors, and we write

$$\mathbf{AB}=\mathbf{CD}.$$

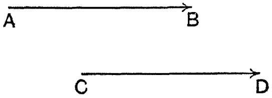

FIG. 67.

But by changing the sense of **CD** we have

$$\mathbf{AB}=-\mathbf{CD}, \quad \text{or} \quad \mathbf{AB}+\mathbf{CD}=0.$$

The connection between sense in geometry and sign in algebra was considered on p. 70.

Notation. In order to avoid the repetition of the word vector, Greek letters will often be used to symbolise them, the corresponding English letter denoting the magnitude; a, b, c, d, e denote the magnitudes of α, β, γ, δ, ϵ.[1] Block letters are often used in books to denote vectors—**A**, **B**, .. —this is the "Heaviside" notation. Owing to the difficulty of *writing* these, small Greek letters are to be preferred—"Henrici" notation. When the vector is denoted by the letters placed at its ends, block letters will be used; in writing, use the Maxwell notation.

Addition of Vectors. The process is exactly the same as for displacements, the formal enunciation is: **To add vectors, place the first anywhere, at the end of the first place the beginning of the second, at the end of the second the beginning**

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[1] For the pronunciation see Note *B*, p. 374.

of the third, and so on; then the vector from the beginning of the first to the end of the last is the sum of the given vectors.

The vector giving the sum is often called the **resultant vector,** and in relation to this the vectors are called the **components.**

When the end of the last and the beginning of the first vector coincide, the sum is zero and the vectors are said to cancel.

(5) Draw any five lines and give them senses. Denoting these lines by α, β, γ, δ, ϵ prove that

$$\alpha+\beta+\gamma+\delta+\epsilon=\beta+\delta+\epsilon+\gamma+\alpha=\delta+\gamma+\beta+\alpha+\epsilon.$$

(6) Find the vector sum of

$$\alpha-\beta+\gamma-\delta+\epsilon=\gamma-\delta-\beta+\epsilon+\alpha.$$

(7) The lengths of five vectors are 3·5, 2·6, 4·7, 6·2 and 7·8 cms. respectively, and they point N., S.W., 20° S. of E., 25° E. of S. and E. Find the resultant vector, taking care to give its magnitude, direction and sense.

Order of Addition. The order in which vectors are added is immaterial.

In Fig. 68,

$$\sigma=\alpha+\beta+\gamma+\delta,$$

but $$\beta+\gamma=\gamma+\beta,$$

$$\therefore\ \sigma=\alpha+\gamma+\beta+\delta.$$

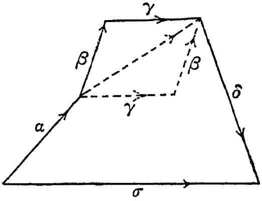

FIG. 68.

By changing the order two at a time, any desired order may be obtained, and hence the theorem is established.

(8) Draw a regular hexagon *OABCDE*, find the vector σ such that $\alpha+\beta+\gamma+\delta+\epsilon+\sigma=0$, where $\alpha=\mathbf{OA}$, $\beta=\mathbf{OB}$, ...

Average Velocity. When a body moves, its displacement, from some standard point or origin, changes with the time.

The total displacement in any time divided by the time is defined as the average velocity for that time.

This average velocity is then measured by a displacement and is therefore a vector quantity.

EXAMPLE. *A train at* 10 *a.m. is* 100 *miles N.E. of London, at noon it is* 80 *miles* 15° *N. of E. What is its average velocity?*

Draw OP_1 (Fig. 69) and OP_2 to scale, giving the displacements; measure P_1P_2 on same scale and find its direction. Bisect P_1P_2 at M. Then $\mathbf{P_1P_2}$ is the total displacement, and P_1M gives the average velocity in miles per hour.

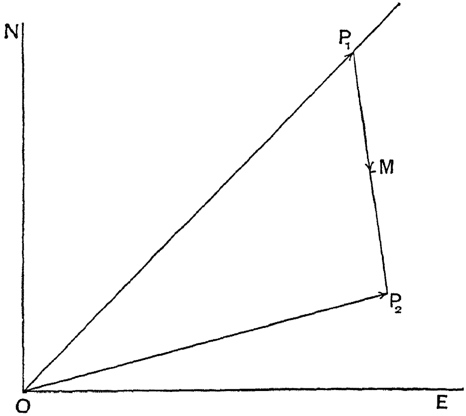

FIG. 69.

It should be evident, that, as nothing is known of the motion between 10 a.m. and noon, the displacement in one hour is not necessarily $\frac{1}{2}\mathbf{P_1P_2}$.

The average velocity means only that velocity which, if it remained constant, would give the actual displacement in the given time.

The average speed of a point for any interval of time is defined as the distance traversed divided by the time.

(9) A man walks from O towards the S.W. for 4 miles and arrives at A in 65 minutes, he then walks for 3 miles W. to B in 40 minutes, then 6·5 miles due N. to C in 105 minutes. What are his average velocities from O to A, O to B, O to C, and what are his average speeds?

(10) A man walks round (clockwise) a rectangular field $ABCD$ in 27 minutes. Starting at A he is at B (due E. of A), distant 200 yds., in 7 minutes; at C, distant 150 yds. from B, in 16 minutes; and at D in 21 minutes. What are his average velocities and speeds from A to B, A to C, A to D and from A back to A?

Speed and Velocity. A point moves on a curve from P (Fig. 70) to P_1 in time t. Its displacement in that time is the chord $\mathbf{PP}_1$ and its average velocity is $\frac{\mathbf{PP}_1}{t}$.

During this time it travels over the distance PP_1 (arc) and $\frac{\text{arc } PP_1}{t}$ is defined as the **average speed.**

If we take t very small, then the arc, the chord, and the tangent at P become indistinguishable the one from the other at P. At the limit, when P_1 approaches nearer and nearer and finally comes up to P, the chord P_1P produced becomes the tangent at P, and the magnitude of the velocity is the speed at P. The direction of motion is therefore always tangential to the path.

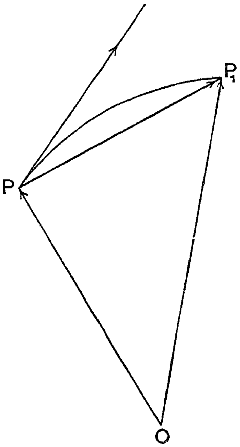

FIG. 70.

At every instant of the motion the point is moving with a definite speed in a definite direction.

The smaller the time interval, the smaller will be the vector $\mathbf{PP}_1$, and the more nearly will the average velocity be the same as at the beginning or end of the time interval. The problem of finding, in particular cases, to what fixed value this average velocity tends, as the interval of time is taken smaller and smaller without limit, belongs to the Calculus and cannot be discussed here. This limiting value of the average velocity is evidently the velocity at the instant under consideration, and its magnitude is the speed of the point at that instant.

A velocity is specified by giving the speed, the direction and the sense of the motion.

Speed is constant only when the point passes over equal distances in equal times, no matter how small the equal times may be, or the average speed is always the same whatever the time interval.

Velocity is constant only when the speed is constant, and the direction and sense of the motion remain unaltered.

A point moving along a curved path may have constant speed but cannot have constant velocity.

Units. The unit of length being a foot, and the unit of time a second, the unit of speed is a ft. per sec. often written 1 ft./sec. Other units in common use are 1 mile per hour, or 1 m./hr., and 1 cm. per sec. or 1 cm./sec.

DEFINITION. **The Velocity of a point is its rate of displacement** and is measured by the displacement in unit time, or the displacement that would have taken place if the velocity had remained constant.

Velocity is, therefore, a vector quantity and can be represented by a line vector; the length of the vector represents to scale the magnitude of the velocity (the speed), the direction and sense of the motion being shown by the direction and sense of the vector.

Velocities are added or compounded like vectors since they are measured by displacements.

EXAMPLE. *A ship is moving due W. at a speed of* 15 *miles an hour; a passenger runs across the deck from S. to N. at* 7 *miles an hour, find the velocity of the passenger relative to the earth.*

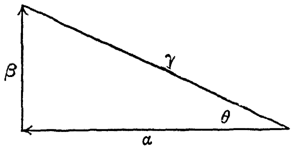

FIG. 71.

Set off α (Fig. 71) from right to left of length 15 cms.; add β of length 7 cms. drawn vertically upwards; then γ the sum ($\gamma = \alpha + \beta$) gives the magnitude, direction and sense of the velocity required. Scale γ in cms. and measure the angle θ.

(11) A ship sails N. relative to the water at 5 ft. per sec. whilst a current takes it E. at 3 ft. per sec.; what is the velocity of the ship relative to the earth?

(12) A river current runs at 2 miles an hour; in what direction should a swimmer go, who can swim 2·5 miles an hour, in order to cross the river perpendicular to the banks? [Draw the vector of the velocity of the current; through the beginning of this, draw a line perpendicular to it, and with the other end as centre, describe a circular arc of radius 2·5 to cut the perpendicular. This construction gives the velocities of the swimmer relative to the water and to the land. There are two solutions, giving the directions from both banks.]

(13) A boat can be rowed at 6 miles an hour in still water, a river current flows at 3 miles an hour; how should the head of the boat be pointed if it be desired to cross the river at an angle of 45° up stream?

(14) A train travels E. at 65 miles an hour, a shot is projected from the train at an angle of 30° with the forward direction and at a speed of 200 miles an hour relative to the train; what is the velocity of the shot relative to the earth?

(15) A ball is moving at 10 miles an hour S.W. and is struck by a bat with a force which would, if the ball had been at rest, have given it a speed of 8 miles an hour due S.; with what velocity does the ball leave the bat?

***Relative Velocity.** A point at any instant can only have one definite velocity; it is impossible to conceive it as moving in two different directions, or with two different speeds, at one and the same instant. Relative to other moving points it may have all sorts of velocities.

*EXAMPLE. *A train moves* 20° *E. of N. at* 50 *miles an hour; another train is moving W. at* 22·6 *miles an hour, and a third is travelling due S. at* 35·2 *miles an hour. What are the velocities of the first and third relative to the second?*

Notice that the relative motions of two or more bodies are unaffected by any motion common to them all; thus, the relative motions of trains, people, ships, etc., are quite independent of the motion of the earth round the sun. We may, therefore, suppose any common velocity given to the trains.

In Fig. 72, α represents the velocity of the first train, β and γ those of the second and third.

Add $-\beta$ to α, then the sum σ_1 gives the relative velocity of the first to the second train, for $-\beta$ reduces the second train to rest, and then $\alpha - \beta$ is the velocity of the first relative to a supposed fixed point. Scale σ_1 and measure the angle it makes with the E. line (61·8 miles an hour 50·3° N. of E.).

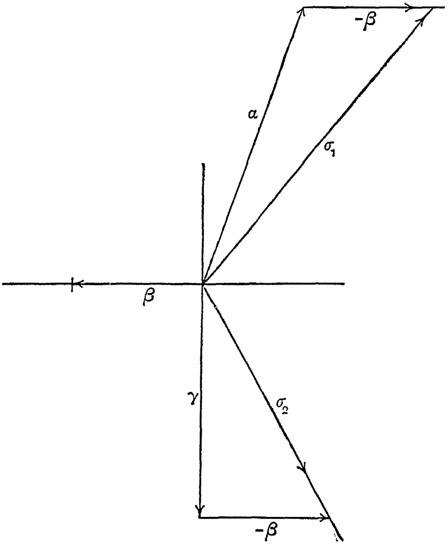

FIG. 72.

Similarly, add $-\beta$ to γ and obtain σ_2 the relative velocity of the third to the second.

(16) Two cyclists meet on a road, one is going S. at 10 miles an hour, the other N. at 12 miles an hour; what are their relative velocities?

(17) A cyclist travels N.W. at 12 miles an hour, the wind is due E. and travels at 20 miles an hour; what is the apparent direction and speed of the wind to the cyclist? [Add to the wind velocity one of 12 miles an hour S.E.]

(18) A train travels due W. at 40 miles an hour; the smoke from its funnel makes an angle of 157° with the forward motion; the wind is blowing from the N., what is its speed?

(19) In the last example if the speed of the wind had been 20 miles an hour, what are the possible directions whence it could have come?

*Example. *Two ships are 20 miles apart, one* (A) *is then due E. of the other* (B), *and is steaming due N. at* 18 *miles an hour;* B *is steaming E. at* 15 *miles an hour. Find when they will be nearest one another, and their distance apart at that time.*

Suppose a speed of 18 miles an hour due S. is given to both, then A is reduced to rest, and the relative motion of A and B is unaltered. Hence, add the vectors (Fig. 73) representing the velocities 15 due E. and 18 due S. and obtain a resultant vector **BC**(γ). From A drop a perpendicular on BC. and measure its length on the scale to which the distance AB was drawn; it is evidently the shortest distance the ships will be apart. **BC** is the relative displacement in one hour; in order to find the time for the relative displacement **BD**, set off $BC_1 = 10$ cms. in any direction to represent 1 hour; then, if DD_1 be drawn parallel to CC_1, BD_1 gives the time in hours.

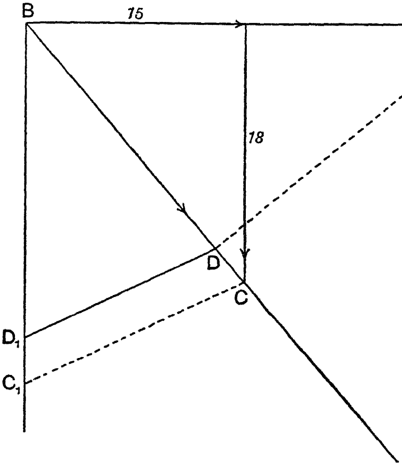

Fig. 73.

*Example. *Find by construction the actual positions of the two steamers, in the last example, when at their shortest distance apart.*

To do this, draw through D a line due North cutting BA

at B_1, then B_1 is the required position of B. Through B_1 draw B_1A_1 parallel to DA cutting the N. line at A in A_1; then A_1 is the required position of A.

(20) Two ships are 11 miles apart, and both are steaming direct towards the same point distant 11 and 7 miles from them respectively. They both travel at 14 miles an hour. Find their shortest distance apart and the corresponding time.

* **Total Acceleration.** When the velocity of a body changes, the motion is said to be **accelerated.** This **acceleration** may be due to the velocity increasing or diminishing (**retardation**), or to the change in the direction of the motion or to both. Thus, if α_1 (Fig. 74) gives the velocity at one instant and α_2 at a subsequent one, the magnitude only has changed, and the **total acceleration** is $\alpha_2 - \alpha_1$ and is negative.

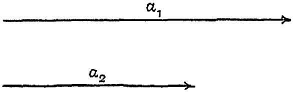

FIG. 74.

If β_1 and β_2 (Fig. 75) denote the velocities at two instants, then the change in the velocity is γ, where

$$\beta_1 + \gamma = \beta_2 \text{ or } \gamma = \beta_2 - \beta_1.$$

γ, the change in the velocity, is simply the velocity which must be added on—as a vector—to the initial velocity β_1 to give the final velocity β_2. Change in velocity is then a vector quantity and must be represented by a vector.

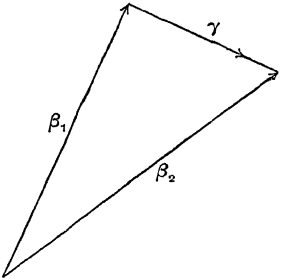

FIG. 75.

The velocity has changed both in magnitude and in direction and γ is the total acceleration.

(21) A cyclist at noon is travelling N. at 12 miles per hour; at 1 p.m. he is travelling 10 miles an hour 75° E. of N.; what is his total acceleration?

* **Average Acceleration.** Dividing the total acceleration by the time we get the **average acceleration.**

* **Acceleration.** Notice that both total and average accelerations are vector quantities, and the latter gives the average velocity added per unit of time.

If the acceleration is constant, *i.e.* if the same velocity be added during equal intervals of time, no matter how small the latter may be, then the velocity added per second is the acceleration.

If the acceleration changes, then the value to which the **average acceleration** approximates, as the time interval becomes smaller and smaller without limit, is the acceleration at the instant (cf. Velocity, on p. 77).

* DEFINITION. **Acceleration is the rate of change of velocity,** and is measured by the velocity added per unit time, or the velocity that would have been added if the acceleration had kept constant.

Acceleration is therefore a vector quantity, and accelerations are added (or compounded) as vectors.

It can be shewn by experiment that bodies falling freely under the influence of gravity have a constant acceleration directed towards the centre of the earth, and measured by a velocity of 32·2 ft. per second added per second, or 32·2 ft. per sec. per sec. (at Greenwich). This acceleration is usually denoted by the letter g.

(22) A train at noon is moving 35° E. of N. with a speed of 32 m./hr. At 12 h. 35 m. p.m. it is moving with a speed of 27 m./hr. 50° E. of N. What are its total and average accelerations during this time?

(23) A point moves in a horizontal circle of radius 5·3 ft. in the contra-clockwise sense. When at the most northern point its speed is 11·3 ft./sec., when at the W. point it is 12·7 ft./sec., and when at the S. and E. points its speed is 14·8 and 11·3 ft./sec. If the time taken to move through each quadrant be 1·2 minutes, find the average and total accelerations for 1, 2, 3 and 4 quadrants.

(24) Two trains are moving towards the same point in directions inclined at 70° with one another. One train is increasing speed at the rate of 33 ft. per sec. per sec.; the other is diminishing its speed at the rate of 17 ft. per sec. per sec. Find the acceleration of the first relative to the second.

Components of a Vector. Finding the sum of a number of vectors is a unique process; *i.e.* one and only one resultant vector is obtained. The converse, *i.e.* finding the components when the resultant is known, is not unique in general. The components of a vector in two given directions are, however, uniquely determined.

EXAMPLE. *Find the components of a in the two given directions.*

From the ends of a (Fig. 76), draw lines parallel to the two given directions, these determine two vectors a_1 and a_2 which are the components. The construction can be done in two ways as indicated, but the component vectors are the same in both constructions.

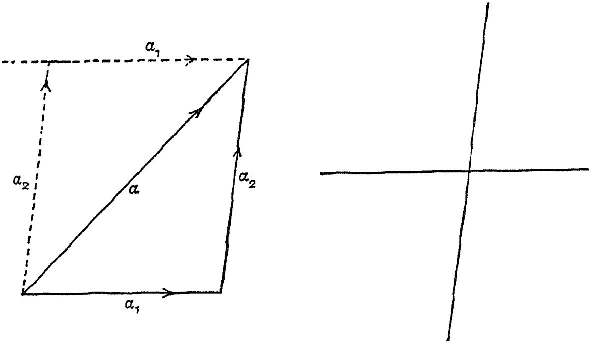

FIG. 76.

(25) Draw any vector a and any three lines; shew that any number of components of a can be found in the three directions.

(26) a represents a velocity of 10 ft. per sec. due N.; find the component velocities N.W. and N.E.

(27) Find the components of a displacement 15 ft. E. in direction, making angles 15° N. and 30° S. respectively with this line.

*(28) A falling stone has an acceleration of 32·2 ft. per sec. per sec. vertically downwards; find the components along and perpendicular to a line making an angle of 60° with the horizontal.

*(29) A train has an acceleration of 5 ft. per sec. per sec. down an incline of 1 in 6 (1 vertical 6 along the incline); find the component accelerations horizontally and vertically.

When only one component is spoken of, the other is supposed to be at right angles to the first component.

(30) A ship journeys 50 miles 20° N. of W.; what is its displacement due W.

*(31) A bead slides freely down a straight wire making 65° with the horizontal; what is the acceleration of the bead down the wire?

*(32) A cable car fails to grip on a down incline of 10 in 73; if the retardation due to friction be equivalent to a negative acceleration of 2·8 ft. per sec. per sec., what is the actual acceleration of the car?

*(33) If in question 32 a man jumps up from his seat so that his body has a vertical acceleration (relative to the car) of 1·9 feet per sec. per sec., what is the real acceleration of his body?

Multiplication of Vectors by Scalars. Multiplying a vector by a number merely multiplies the length of that vector, thus $n\alpha$ means a vector n times as long as α.

Similarly multiplying a vector by any scalar quantity multiplies its length by that quantity.

OAB (Fig. 77) is any triangle. If A_1B_1 be drawn parallel to the base AB, cutting OA and OB produced in A_1 and B_1, then we know that OAB and OA_1B_1 are similar triangles. Hence A_1B_1 is the same multiple of AB that OA_1 is of OA.

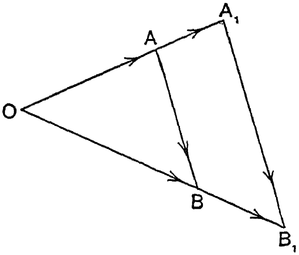

FIG. 77.

If $\mathbf{OA}=\alpha$ and $\mathbf{OB}=\beta$, be the two sides of the triangle,
then if $\mathbf{OA}_1=n\alpha$ and $\mathbf{OB}_1=n\beta$
we have $\mathbf{AB}=\beta-\alpha$ and $\mathbf{A}_1\mathbf{B}_1=n(\beta-\alpha)$,
so that from $\mathbf{A}_1\mathbf{B}_1=\mathbf{OB}_1-\mathbf{OA}_1$,
we have the vector law,

$$n(\beta-\alpha)=n\beta-n\alpha, \qquad \text{(1)}$$

(34) Establish the equation $n\alpha+n\beta=n(\alpha+\beta)$ by (i) adding to the sum of n vectors α the sum of n vectors β, and (ii) adding β to α and then adding n of these vectors together.

Centre of Mean Position. Given two points A and B, the point M bisecting the line AB evidently occupies a mean position with regard to A and B.

Choose any two origins O and O_1 (Fig. 78), let **OA** $=\alpha$ and **OB** $=\beta$, and from O_1 draw α and add β to it, then the sum

$$\alpha+\beta=\sigma;$$

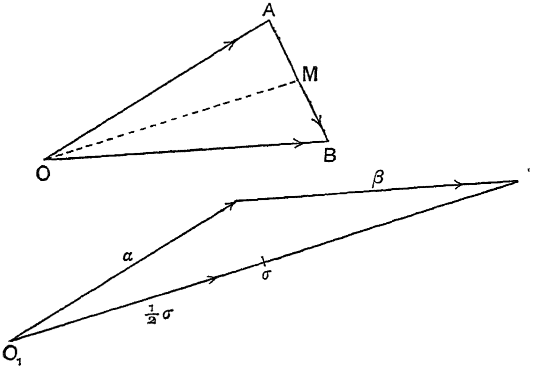

FIG. 78.

from O set off $\frac{1}{2}\sigma$ and shew that the point M thus determined is the mid-point of AB. (This is also obvious from OAB, for **AB** $=\beta-\alpha$, and, therefore, the vector to the mid-point of AB is

$$\alpha+\tfrac{1}{2}(\beta-\alpha)=\frac{\alpha+\beta}{2}=\tfrac{1}{2}\sigma,$$

proving, incidentally, that the diagonals of parallelograms bisect one another.)

Take any three points A, B, C (Fig. 79), and a point of reference O. Find the sum $\alpha+\beta+\gamma$ ($=\sigma$) as indicated (away from A, B, C). From O set off $\frac{1}{3}\sigma$ and determine thus the point M, the centre of mean position.

Instead of O take other origins O_1 and O_2 and shew by a similar construction that the same point M is obtained. This shews that M depends on A, B and C, and not on the origin used to determine it. Draw the medians of ABC and see that M is their point of concurrence.

(35) Draw a regular hexagon, take any origin (not the centre) and find in a separate figure the sum of the six position vectors of the vertices. Set off from the origin $\frac{1}{6}$ of this sum, and thus find M, the centre of mean position of the vertices.

Perform a similar construction when the origin is at the centre.

(36) Draw a parallelogram and find the centre of mean position of the vertices by taking the origin (i) outside, (ii) at the intersection of the diagonals.

In each case the centre of mean position is the end point of the vector, drawn from the origin, which is $\frac{1}{4}$ of the sum of the position vectors of the vertices.

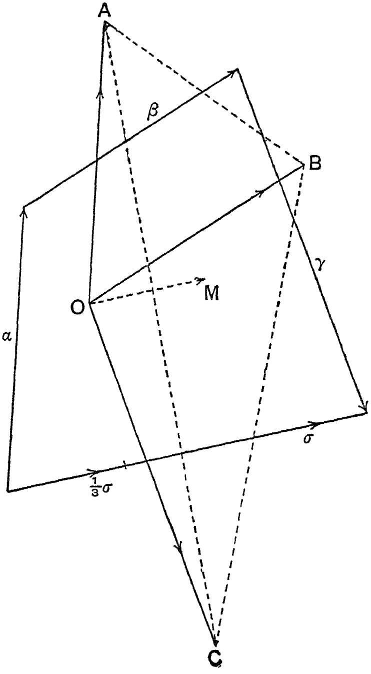

Fig. 79.

(37) Take any five points and any origin, find $\frac{1}{5}$ of the sum of the position vectors; set this off from the origin and determine thus the point M. Choose another origin and show, by a similar construction, that the same point M is determined.

Definition. **Generally, if there are n points $A, B, C, \ldots$ whose position vectors with reference to some origin O are $\alpha, \beta, \gamma, \ldots$, then $\frac{\alpha+\beta+\gamma+\ldots}{n}$ is the position vector of a point M called the centre of mean position.**

Let α, β, γ, ... (Fig. 80) be the position vectors of the points relative to O, α_1, β_1, γ_1, ... their position vectors relative to O_1, and let $\rho = \mathbf{OO_1}$, then

$$\mathbf{OM} = \frac{\alpha + \beta + \gamma + \dots}{n} \text{ and } \mathbf{O_1M'} = \frac{\alpha_1 + \beta_1 + \gamma_1 + \dots}{n}.$$

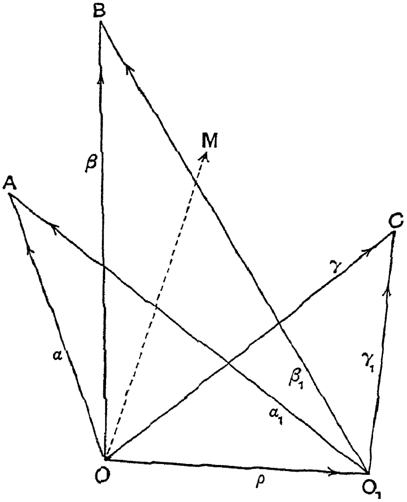

FIG. 80.

But $\qquad \alpha_1 = -\rho + \alpha, \; \beta_1 = -\rho + \beta, \dots .$

$$\therefore \; \mathbf{O_1M'} = \frac{\alpha + \beta + \gamma + \dots - n\rho}{n} = \mathbf{OM} - \rho;$$

$$\therefore \; M' \text{ is at } M.$$

The centre of mean position is thus a point dependent only on the relative position of the points themselves and not on the origin used to determine it.

The points A, B, C, ... need not be in a plane.

(38) The coordinates of five points are (1, 1·2), (−1, 1·5), (2·1, 1·3), (3·3, −1·4) and (−2, −3·4); find the centre of mean position. (Add the vectors and divide by 5; set off this one-fifth vector from the origin, and measure the coordinates of its end point M.)

Centre of Figure. In the case of a line, curve, area or volume, the centre of mean position of all the points in it is called the **centre of figure** or **centroid.** In many cases we can determine the centre of figure by inspection.

A straight line. Choosing the origin at the point of bisection we see that to every point P, having a position vector ρ, there is a point P_1, having $-\rho$ for its vector, hence, $\Sigma\rho = O$, and O is the centre of mean position. Where is the centre of figure for lines bounding a square, a rectangle, a parallelogram, a circle, respectively, and why?

Area of a Parallelogram. Taking the origin at the intersection of the diagonals, to every point P or Q (Fig. 81) there is a corresponding point P_1 or Q_1 such that $\mathbf{OP} + \mathbf{OP}_1$ or $\mathbf{OQ} + \mathbf{OQ}_1 = 0$, and, therefore, the centre of mean position of figure is at this point of intersection.

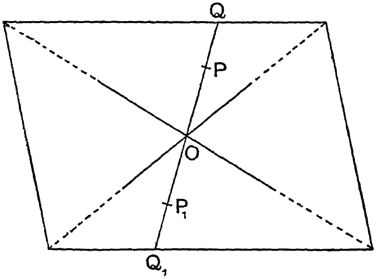

FIG. 81.

(39) Where is the centre of figure of the area of a rectangle, of a circle, of a regular hexagon?

(40) Shew that the centre of figure of a regular pentagonal area cannot be proved to be at the centre of the circumscribing circle by this argument alone.

(41) Mark the positions of any seven points on your drawing paper. Find the centre of mean position of any four of them and then of the remaining three, and mark these two points. Find the centre of mean position of these two points, counting the first one four times and the last three times. Is this final centre of mean position the same as would be determined directly from the position vectors of the seven points?

Theorem on Centres of Mean Position. **In finding the centre of mean position of a system of points, any number of them may be replaced by their centre of mean position, if the position vector of that point be multiplied by that number.**

Let there be m_1 points whose position vectors are $\alpha_1, \beta_1, \gamma_1, \dots$.
" " m_2 " " " " $\alpha_2, \beta_2, \gamma_2, \dots$.
" " m_3 " " " " $\alpha_3, \beta_3, \gamma_3, \dots$.

Then the position vector σ of the centre of mean position for the whole $m_1+m_2+m_3$ points, is given by

$$\begin{aligned}(m_1+m_2+m_3)\sigma &= \Sigma(\alpha_1+\alpha_2+\alpha_3)\\ &= \Sigma\alpha_1+\Sigma\alpha_2+\Sigma\alpha_3.\end{aligned}$$

If σ_1 is the centre of mean position for the m_1 points, σ_2 that for the m_2 points and σ_3 that for the m_3 points, then

$$m_1\sigma_1=\Sigma\alpha_1,\ m_2\sigma_2=\Sigma\alpha_2 \text{ and } m_3\sigma_3=\Sigma\alpha_3;$$

$$\therefore\ (m_1+m_2+m_3)\sigma = m_1\sigma_1+m_2\sigma_2+m_3\sigma_3.$$

The m_1 points may therefore be treated as if concentrated at their centre of mean position, provided the position vector of the latter be counted m_1 times, and so on for the other points.

Evidently the argument holds however many partial systems of points we suppose the whole system divided up into.

Mass-Centres. Let there be n points having unit mass at each point, then the centre of mean position of the points is called the **mass-centre** of the masses. If $\alpha_1, \alpha_2 \dots$ are the position vectors of the points, and σ that of the mass-centre, then

$$n\sigma=\Sigma\alpha_1.$$

Divide the points up into groups, m_1 points having mass-centre σ_1, m_2 points mass-centre σ_2, ... then, since $n=\Sigma m_1$, we have

$$(m_1+m_2+m_3+\dots)\sigma = m_1\sigma_1+m_2\sigma_2+\dots. \qquad \dots\dots\dots(1)$$

This equation remains unaltered however the m_1 points be moved, provided that their mass-centre given by σ_1 remains unaltered; we may, therefore, suppose them to come together and coincide at σ_1, and so for the other partial systems.

At the end point of σ_1 we have now a mass m_1 units, at σ_2 a mass m_2 units .... Moreover, equation (1) remains true when multiplied throughout by the same scalar quantity, and hence $m_1, m_2, \ldots$ may be taken as the masses at the points.

Hence, **given a number of points $M_1, M_2, \ldots$ having masses $m_1, m_2, \ldots$ concentrated there, and having position vectors $\sigma_1, \sigma_2, \ldots$ the mass centre of the system is given by the position vector σ, where**

$$\sigma \Sigma m_1 = \Sigma m_1 \sigma_1 .$$

Mass-Points or Particles. We have thus arrived at the conception of points with masses concentrated at them; such points are called **mass-points**, and have exactly the same meaning as the more usual term **particles.**

To find the Mass-Centre (M.C.) **of a number of mass-points**: Choose any convenient origin O, multiply each position vector by the mass at its end point, add all these mass-vectors, and divide the resultant mass-vector by the sum of the masses. Set off from O the vector so determined; its end point will be the M.C. required.

EXAMPLE. *Find the Mass-centre* (M.C.) *of three particles of masses* 1, 2 *and* 3 *grammes placed at the vertices of an equilateral triangle.*

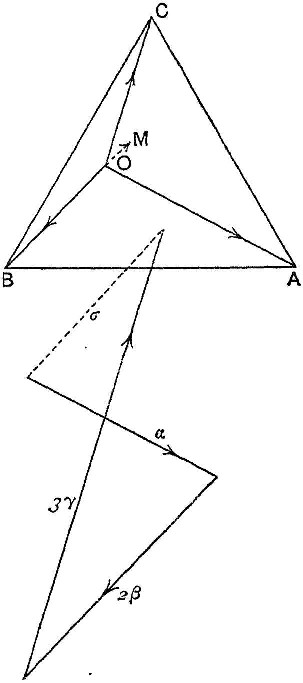

FIG. 82.

Draw any equilateral triangle ABC (Fig. 82). Take any origin O. Draw α equal to **OA**, add 2β (where $\beta =$ **OB**), and 3γ (where $\gamma =$ **OC**). Then, if $\sigma = \alpha + 2\beta + 3\gamma$, set off $\frac{1}{6}\sigma$ from O, and find M the M.C. required.

(42) Find the M.C. of four particles of masses 1, 2, 3, 4 grammes, placed at equal intervals round a circle of radius 3 inches.

(43) Find the M.C. of five particles of masses, 2, 3, 1·8, 3·3, and 4·7 lbs. placed in order at the vertices of a regular pentagon of 1·5″ side.

(44) Take the M.C. as found in Ex. 43 as origin, repeat the construction, and show that the vector polygon is closed, *i.e.* that the origin is the M.C.

Mass-Points in a Line. For two points, A and B, having masses 3 and 2 lbs., we have, by taking the origin O in the line (Fig. 83),

$$3\mathbf{OA} + 2\mathbf{OB} = 5\mathbf{OM}.$$

If M be now taken as origin,

$$3\mathbf{MA} + 2\mathbf{MB} = 0,$$

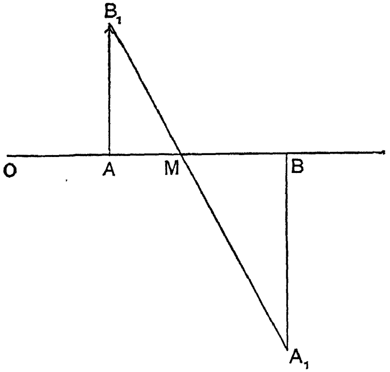

FIG. 83.

and therefore M divides AB inversely as the masses. Hence the simplest construction would be: Set off $BA_1 = 3''$, and AB_1 parallel to BA_1 and of length $2''$. Put a straight edge along A_1B_1, and mark the point M where it cuts AB.

(45) At A and B, distant apart 3″, are masses given by lines of length 7·1 and 5·6 cms. Find the M.C.

(46) If in Ex. 45 the masses at A and B are given by (i) the squares, (ii) the cubes of the lengths, determine graphically in each case the position of the M.C.

(47) Draw any line and mark an origin O and four points in the line. Suppose masses of 2, 3, 1, and 4 grammes to be at the points. Construct the position of the M.C., and measure its distance from O. Shew that this position could have been calculated by multiplying each mass by its distance from O (positive if to the right, negative if to the left), adding the products together, and dividing by 10, the sum of the masses.

Formula for the M.C. of Points in a Line. Let $x_1, x_2, x_3, \ldots$ be the distances of a number of points in a line from an origin O in that line, and $m_1, m_2, m_3, \ldots$ the masses at those points.

Then, all the position vectors being parallel, they are added as scalars, one sense giving a positive scalar, the opposite a negative one. If, then, $\bar{x}$ denote the distance of the M.C. from O, we have

$$\bar{x}\Sigma m_1 = m_1x_1 + m_2x_2 + \ldots = \Sigma m_1x_1.$$

If this sum is negative, it shows that the M.C. lies on the negative side of the origin.

(48) Find by calculation the position of the M.C. of masses 2·7, 3·6, 4·7 and 6·9 grms. situated at points in a line distant 11·7, 1·6, 1·2 and 9·3 cms. from an origin O in the line.

(49) Calculate the position of the M.C. of masses 10, 5, 3, 8, and 1 lbs. situated in a line, the position of the points from a fixed origin in the line being 1, 5, −2, −3, and 4 ft. respectively.

Graphical Construction.

EXAMPLE. *Masses given by lines* m$_1$, m$_2$, m$_3$ *and* m$_4$ *are concentrated at points* X$_1$, X$_2$, X$_3$, X$_4$ *in a line, to construct the position of the mass-centre.*

Draw $m_1, m_2, \ldots$ and the distances $X_1, X_2 \ldots$ twice the size of those in Fig. 84.

Through any point O in the line draw Oy perpendicular to it, set off from O along this perpendicular OM_1, OM_2, OM_3, OM_4 and OM equal to m_1, m_2, m_3, m_4, and $m_1+m_2+m_3+m_4$ $(=\Sigma m_1)$. OM should be found by the strip method of addition.

Through M_1 draw M_1x_1 parallel to MX_1.
,, M_2 ,, M_2x_2 ,, MX_2.
,, M_3 ,, M_3x_3 ,, MX_3.
,, M_4 ,, M_4x_4 ,, MX_4.

By the strip method find the sum of Ox_1, Ox_2, Ox_3, Ox_4, and set off OX equal to this sum, then X is the M.C. of the four mass-points.

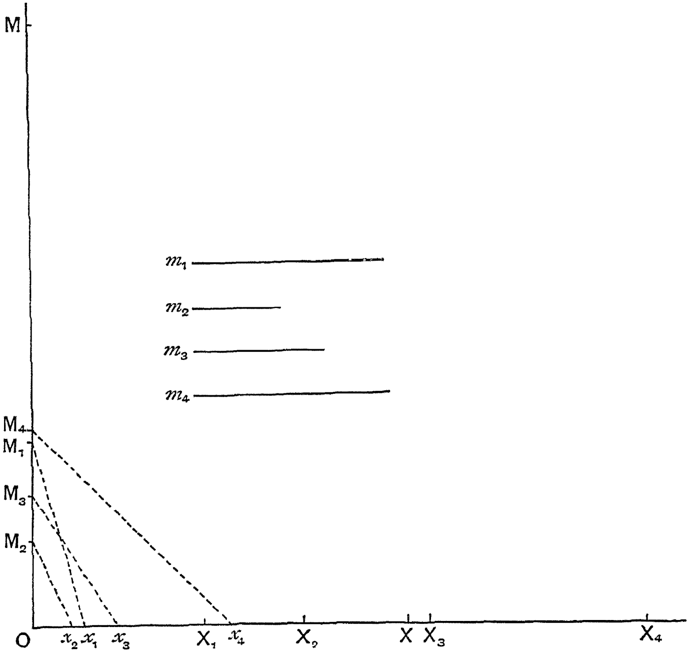

FIG. 84.

Proof.

$$\frac{OM}{OM_1} = \frac{OX_1}{Ox_1};$$

$$\therefore\ OM \cdot Ox_1 = OM_1 \cdot OX_1.$$

Similarly,

$$OM \cdot Ox_2 = OM_2 \cdot OX_2,$$
$$OM \cdot Ox_3 = OM_3 \cdot OX_3,$$
$$OM \cdot Ox_4 = OM_4 \cdot OX_4;$$

$\therefore$ by addition $OM \cdot \Sigma Ox_1 = \Sigma OM_1 \cdot OX_1$.

The right-hand side of this equation is the sum of the products of each mass and its distance from O, the left-hand side is OX multiplied by the sum of the masses, hence OX is the distance of the M.C. from O.

(50) Repeat the construction as above, drawing Oy through M_1, M_2 and M_4 in turn. Notice that this simplifies the work, since only three parallels have to be drawn.

(51) Masses given by lines of lengths 2·3, 5, 7·8, 8·5 cms., scale 1″ to 5 lbs., are at points (in a straight line) whose distances apart are 0·5, 1, 1·7, 0·8 inches taken in order. Find the M.C. graphically, and test by calculation from the formula of p. 93.

Non-Collinear Mass-Points. For points not in a line and having masses given by lines, general constructions are given (i) on page 112, (ii) on page 299.

A few simple cases may be treated by repeated constructions similar to that on page 92.

(52) Find the M.C. of three masses given by the lines m_1, m_2, m_3, situated at A, B, C respectively, where $AB=5{\cdot}4$, $BC=7{\cdot}5$ and $CA=5{\cdot}98$ cms. First, find the M.C. G_1 of m_1 and m_2, and then the M.C. G_2 of m_3 at C, and m_1+m_2 at G_1.

(53) Masses given by lines of lengths 4·7, 2·3 and 3·8 cms. are situated at points whose coordinates in inches are (1, 0), (2, 3), (3, −1). Find the mass-centre by construction, and give its coordinates.

Mass-Centre of a figure with an axis of Symmetry. *If any curved or broken zig-zag line has an axis of symmetry the mass centre of the line must lie on that axis.*

For to every point P distant MP from the axis of symmetry there is a point P_1 at the same distance on the other side of the axis (Fig. 85). Choosing any origin O, then, for two such points,

$$\begin{aligned} \dot{OP} + \dot{OP_1} &= \dot{OM} + \dot{MP} + \dot{OM} + \dot{MP_1} \\ &= 2\dot{OM}, \end{aligned}$$

and as this holds for every pair of points, the M.C. must lie on the axis.

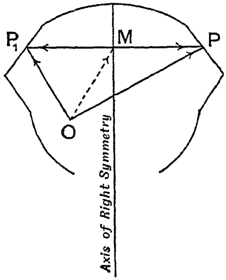

FIG. 85.

When the axis of symmetry is perpendicular to the lines joining corresponding points, it is called an axis of **right symmetry**, otherwise it is an axis of **skew symmetry**. The line joining the mid-points of opposite sides of a rectangle is an axis of right symmetry, a similar line in a parallelogram is an axis of skew symmetry.

M. C. of any number of equal consecutive lines inscribed in a circle. Draw a circle of radius 4″ (Fig. 86), and set off in it six chords, each of length 1·5″, forming an open polygon $A_1A_2A_3A_4A_5A_6A_7$. Draw the axis of symmetry OA_4, O being the centre of the circle, and join A_1A_7.

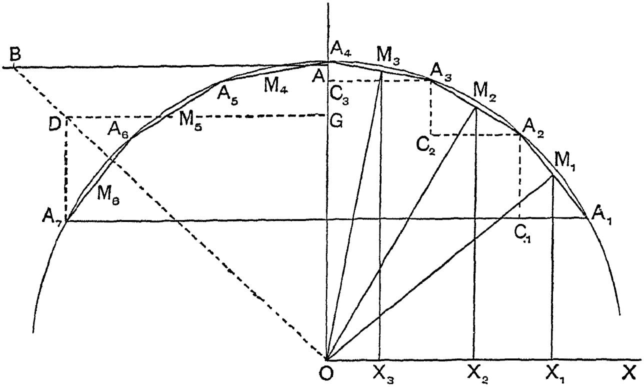

FIG. 86.

Bisect A_1A_2 at M_1 and set off OA along the axis of symmetry equal to OM_1. Draw AB parallel to $A_1A_7 = \frac{1}{2}$ the sum of the sides. Join OB, and draw A_7D parallel to OA, and DG parallel A_1A_7, as in Fig. 86, then the point G so determined is the mass-centre of the six lines.

Proof. Draw through O, OX parallel to A_1A_7, and through A_1 and A_2 lines parallel to OA and OX as in Fig. 86. Through the mid-points M_1, M_2, M_3 of the lines draw M_1X_1, etc., parallel to OA.

Then, by construction, OM_1X_1 is similar to $A_1A_2C_1$;

$$\therefore \frac{OM_1}{M_1X_1} = \frac{A_1A_2}{A_1C_1};$$

$$\therefore A_1A_2 . M_1X_1 = OM_1 . A_1C_1.$$

In the same way $A_2A_3 . M_2X_2 = OM_2 . A_2C_2$,

$$A_3A_4 . M_3X_3 = OM_3 . A_3C_3.$$

But $A_1A_2 = A_2A_3 = A_3A_4$ and $OM_1 = OM_2 = OM_3$;

$\therefore$ by addition $A_1A_2(M_1X_1 + M_2X_2 + M_3X_3) = \frac{1}{2}A_1A_7 . OM_1$.

Now, for finding the mass-centre, we may suppose the mass of each line concentrated at its mid-point, and since OA is the axis of symmetry, the pairs of mass-points M_1, M_6 ... have their mass-centres on OA, and at distances M_1X_1, ... from OX.

But the distance $\bar{y}$ of the M.C. from O is given by

$$A_1A_2(M_1X_1 + M_2X_2 + M_3X_3) = 3A_1A_2 . \bar{y};$$

$$\therefore\ 3A_1A_2 . \bar{y} = \tfrac{1}{2}A_1A_7 . OM_1; \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\therefore\ AB . \bar{y} = GD . OA,$$

or
$$\frac{\bar{y}}{OA} = \frac{GD}{AB},$$

and
$$\bar{y} = OG.$$

If there had been eight sides instead of six, equation (i) would have been $\qquad 4A_1A_2 . \bar{y} = \frac{1}{2}A_1A_9 . OM_1$,

if $2n$ sides, $\qquad nA_1A_2 . \bar{y} = \frac{1}{2}A_1A_{2n+1} . OM_1$,

or $\qquad$ semi-perimeter . $\bar{y}$ = semi-closing chord . perpendicular from centre on polygon,

or $\qquad$ perimeter . $\bar{y}$ = closing chord . perpendicular. (ii)

(54) Find by this method the M.C. of any six sides of a regular heptagon.

Mass-Centre of a Circular Arc. The formula embodied in equation (ii) is independent of the number of sides to the polygon. When the number of sides becomes very large, the sides themselves being very small, the polygon becomes nearly the same as the circular arc, and the perpendicular OM_1 becomes nearly the radius of the circle. The limiting case, when the number of sides becomes infinitely large and their size infinitely small, is the arc itself, and hence for any circular arc

perimeter . $\bar{y}$ = closing chord . radius.

Construction for the M.C. of a Circular Arc. Draw a circular arc BCD (Fig. 87) of radius 4″ and angle 135°, and its axis of symmetry OC. Construct the tangent at C, and step off

$$CD' = \text{arc } CD$$

along it. Join OD', and draw DL and LG parallel and perpendicular to OC, then G is the mass-centre.

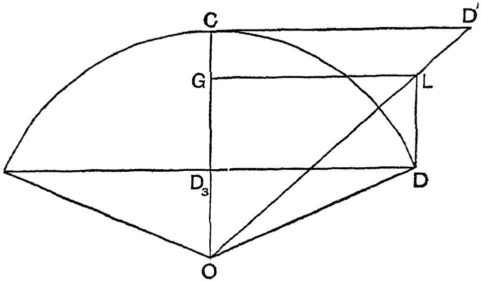

FIG. 87.

Proof. This follows at once from the equation

$$\frac{\bar{y}}{\text{radius}}=\frac{\text{closing chord}}{\text{perimeter}}.$$

From this result a formula for calculation can be deduced, for if r = radius and $2a$ the angle BOD,

we have
$$\frac{\bar{y}}{r}=\frac{r\sin a}{ra};$$

$$\therefore\ \bar{y}=r\frac{\sin a}{a},$$

which, in the case of a semicircular arc, becomes

$$\bar{y}=\frac{2r}{\pi}.$$

(55) Construct the M.C. of a semicircular arc, and compare the measured $\bar{y}$ with the calculated value $\frac{2r}{\pi}$.

(56) Construct the M.C. of a circular arc subtending 270° at the centre.

(57) Find the M.C. of the lines bounding a circular sector of angle 75°.

(58) Find the M.C. of a uniform U-rod formed of two parallel pieces, each of length 6″, connected by a semicircular piece of radius 3″.

Mass-Centres of Areas. If mass be supposed distributed uniformly over an area, the preceding processes enable us to find the mass-centre (or centroid) in many cases.

The principles of general use are:

(i) If the area has an axis of symmetry, the M.C. must lie on that axis.

(ii) If the area can be divided into parts, for each of which the M.C. can be seen by inspection or easily found, then the M.C. of the whole is found by finding the M.C. of these points, each point having a mass proportional to the corresponding area.

EXAMPLE. *To find the* M.C. *of a triangular lamina* ABC.

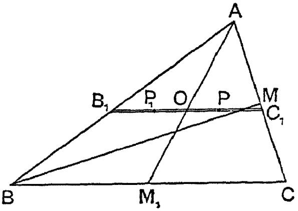

FIG. 88.

Draw the median AM_1 (Fig. 88); this bisects all lines parallel to BC, *i.e.* the M.C.'s of all lines (or very narrow strips) parallel to BC lie in AM_1. Hence to a mass m at P_1 there is an equal mass m at P, where P_1OP is parallel to BC and $\mathbf{OP}_1 + \mathbf{OP} = 0$. AM_1 is an axis of skew symmetry; BC is called the **conjugate** direction.

Draw a second median BM; the point of intersection is the M.C.

(59) Shew, from the property of the M.C., that the three medians meet in a point.

(60) Shew that the triangle and three equal masses placed at the vertices have the same axes of skew symmetry. Hence prove that M.C. of the triangle lies on the median M_1A, $\frac{1}{3}$ of the way up from base.

EXAMPLE. *Find graphically the* M.C. *of the area in Fig.* 89.

Draw the figure to scale.

Find the mass-centres G_1 and G_2 of the two rectangles as indicated. Join G_1G_2, and divide G_1G_2 in the ratio of 16 to 9.

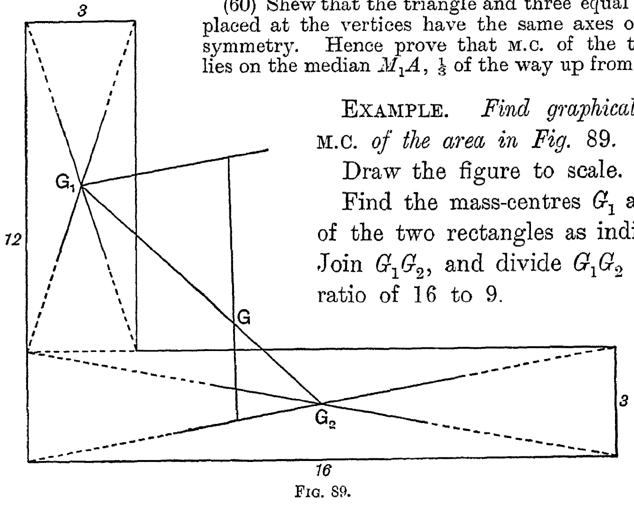

FIG. 89.

Quadrilateral.

EXAMPLE. *Find graphically the* M.C. *of a quadrilateral.*

Draw any quadrilateral $ABCD$ (Fig. 90) and its diagonals AC and BD intersecting in O.

Cut off $AE=CO$. Find the M.C. of BED, it is the same point as the M.C. of $ABCD$.

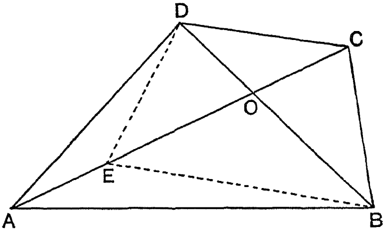

FIG. 90.

Proof. Since $AE=CO$, the medians through D of CAD and OED are coincident. But the M.C. of a triangle (Ex. 60) lies $\frac{1}{3}$ up the median from the base, hence the M.C. of OED is that of CAD.

Similarly, the M.C. of OEB is that of CBA. Also, the masses of AED and BEA, DOC and BOC, DEO and BEO are proportional to their altitudes, which are proportional to DO and OB,

i.e.
$$\frac{ABC}{ACD}=\frac{BO}{DO}=\frac{BEO}{DEO},$$

hence the M.C. of $ABCD$ is that of BED.

EXAMPLE. *Find the* M.C. *of the area of the re-entrant quadrilateral* ABED (Fig. 90).

Set off $AE_1=EO$ along EA produced, and find the M.C. of the triangle DE_1B; it is the M.C. of the re-entrant quadrilateral.

Construct also the M.C. by finding separately the M.C.'s of ADE and ABE.

* EXAMPLE. *Find the* M.C. *of a cross quadrilateral $ABED$.*

Draw a cross quadrilateral from the re-entrant quadrilateral (Fig. 90) by taking E on the other side of AD, and follow out the construction just given; the M.C. of the triangle DE_1B is the M.C. of the cross quadrilateral.

Proof. The cross quadrilateral has an area $ABE - ADE$; the M.C. of ABE is that of E_1BO and the M.C. of ADE is that of E_1DO. The areas added are in each case proportional to the original areas and hence the M.C. of $ABE - ADE$ is that of $E_1BO - E_1DO$, *i.e.* is the M.C. of E_1DB.

(61) Find the mass-centres of the areas of the following figures (Figs. 91–101).

The figures must be drawn full size according to the dimensions given. The angle θ in Fig. 101 is equal to the same lettered angle in Fig. 100.

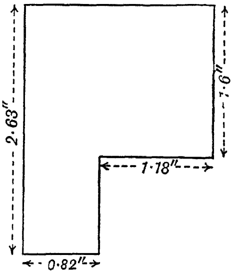

FIG. 91.

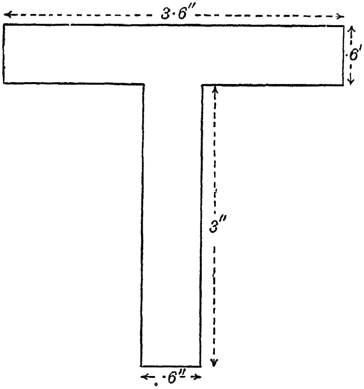

FIG. 92.

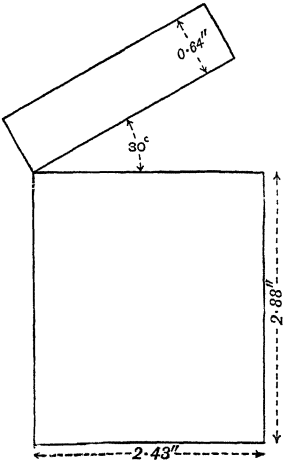

FIG. 93.

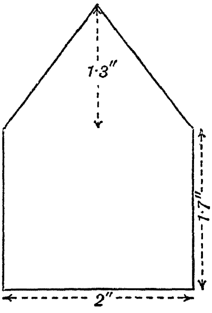

FIG. 94.

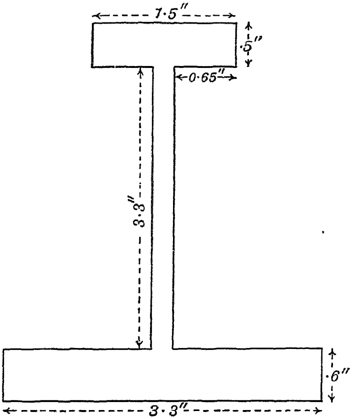
1·5″
·5″
0·65″
3·3″
·6″
3·3″

FIG. 95.

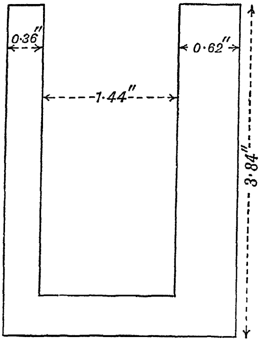
0·36″
0·62″
1·44″
3·84″

FIG. 96.

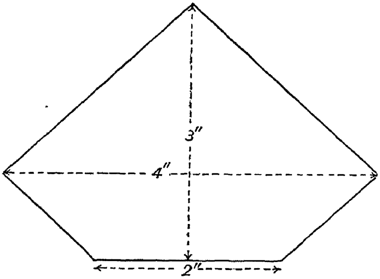

FIG. 97.

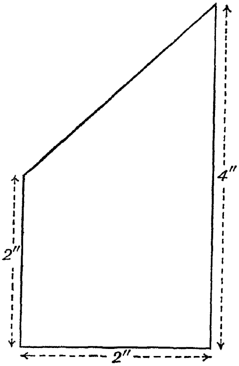

FIG. 98.

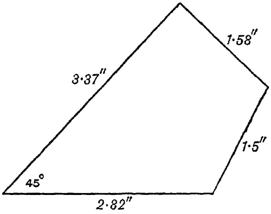
1·58″
3·37″
1·5″
45°
2·82″

FIG. 99

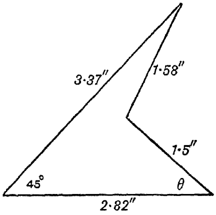
3·37″
1·58″
1·5″
45°
θ
2·82″

FIG. 100.

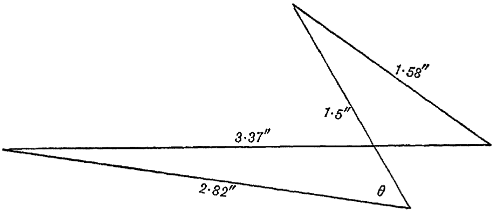
1·58″
1·5″
3·37″
2·82″
θ

FIG. 101.

M.C. of Trapezoidal Area. Draw any trapezium $ABCD$ (Fig. 102) where BC and DA are the parallel sides.

Produce DA to A_1 and BC to B_1 in opposite senses, so that $AA_1 = BC$ and $CB_1 = AD$.

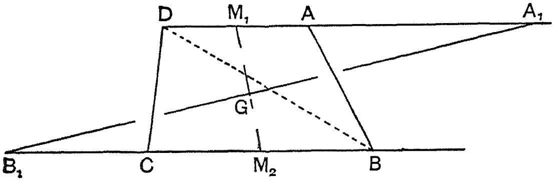

FIG. 102.

Find the point of intersection G of the line A_1B_1 and the line M_1M_2 joining the mid-points of the parallel sides AD and BC.

G is the M.C. of the area.

Proof. The triangles DBC, ABD being of equal altitude have areas proportional to their bases BC and AD. We may, therefore, replace the two triangles by mass-points $\frac{BC}{3}$ at D, B and C and $\frac{AD}{3}$ at A, D and B.

$\therefore$ in the line A_1D there is a mass given by $\frac{BC + 2AD}{3}$.

" " BB_1 " " " $\frac{AD + 2BC}{3}$.

But the M.C. must lie on M_1M_2,

$\therefore$ G, the mass-centre, must divide M_1M_2 so that

$$\frac{GM_1}{GM_2} = \frac{AD + 2BC}{BC + 2AD} = \frac{BC + \frac{1}{2}AD}{AD + \frac{1}{2}BC} = \frac{M_1A_1}{M_2B_1}.$$

Notice that **the distance of the mass-centre of a number of points from a given line is unaltered by any movement of the points parallel to that line.**

(62) Find the M.C. of the trapezium for which $BC = 4{\cdot}3$, $AD = 6{\cdot}1$, $AB = 5{\cdot}8$ and $CD = 4{\cdot}7$ cms., (i) by this method and (ii) by the general quadrilateral method.

(63) Divide the trapezium into a triangle and parallelogram, and find the M.C. of the whole by finding in what ratio it divides the line joining the M.C.'s of the triangle and parallelogram.

M.C. of a Circular Sector.

EXAMPLE. *Construct the* M.C. *of a circular sector of radius* 4 *inches and angle* 120°.

Draw the axis of symmetry OC (Fig. 103).

Set off $OB_1 = \frac{2}{3}OB$ and draw the concentric arc A_1B_1.

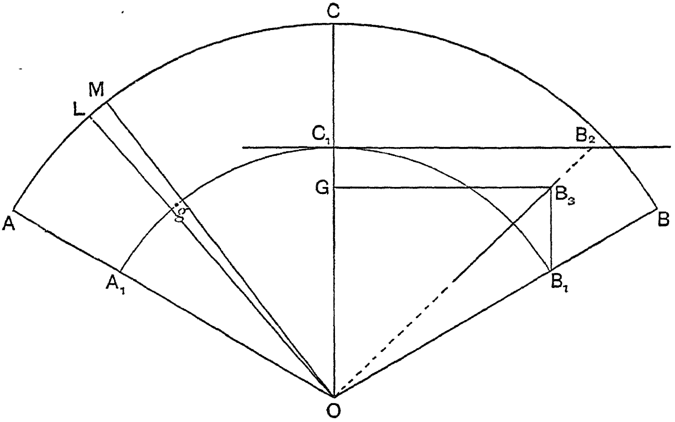

FIG. 103.

Step off the arc C_1B_1 along the tangent C_1B_2; draw B_1B_3 vertically to cut OB_2, and B_3G horizontally to cut OC at G. G is the M.C.

Proof. Suppose the sector divided into a great number of very small sectors, of which OLM is a very enlarged copy, then OLM is at its limit a triangle, LM being a tangent to the circle, the M.C. of this triangle is at g, where $Og = \frac{2}{3}OL$. The sector $OACB$ may therefore be replaced, as far as the M.C. is concerned, by a circular arc of radius $\frac{2}{3}OA$ (p. 97).

The $\bar{y}$ for the circular sector is, therefore, given by

$$\bar{y} = \tfrac{2}{3}r\frac{\sin\alpha}{\alpha},$$

which, in the case of a semicircle, becomes $\dfrac{4r}{3\pi}$.

(64) Find the M.C. of a semicircle and compare your result with the calculated position.

(65) Construct directly from the semicircle the position of the M.C. of a quadrant of a circle, and from that the M.C.'s of $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{8}$ of a quadrant.

Negative Mass and Area.

EXAMPLE. *To find the* M.C. *of the area* $ABCDD_1C_1B_1A_1$, *given that* $AB = 6''$, $BC = 8''$, $AA_1 = 1''$, $A_1B_1 = 5''$, $B_1C_1 = 5''$.

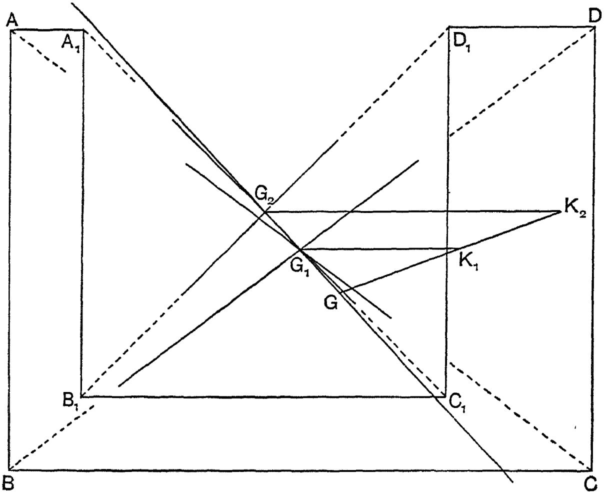

FIG. 104.

Find the mass-centres G_1 and G_2 (Fig. 104) of the rectangles $ABCD$ and $A_1B_1C_1D_1$. Join G_1G_2 and produce. Set off along parallel lines G_2K_2 proportional to $ABCD$, and G_1K_1 proportional to $A_1B_1C_1D_1$ both in the same sense. Where K_1K_2 cuts G_1G_2 is the point G, the mass centre of the area.

Since the given area is the difference between two rectangles, the smaller rectangle (a square) must be considered as a negative area or as having negative mass, and hence G_2K_2, G_1K_1 instead of being set off in opposite senses must have the same sense.

Proof. In order to justify the construction on p. 106 refer to Fig. 105. Let M denote the mass of the whole area enclosed by the outer boundary, and m the mass of the shaded area enclosed by the inner curve.

Let G_1 and G_2 be the mass-centres of the area between the two curves $(M-m)$ and of the shaded area. Then, to find G the M.C. of the whole area M, we have to divide G_1G_2 at G so that

$$\frac{G_2G}{G_1G}=\frac{M-m}{m}.$$

The Graphical construction for effecting this division is indicated in Fig. 105.

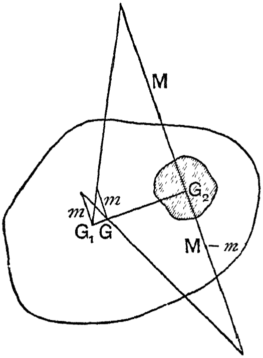

FIG. 105.

On the other hand, if G and G_2 be known, G_1 can be determined.

Add unity to each side of the last equation; then since

$$\frac{M-m}{m}+1=\frac{M}{m},$$

we get
$$\frac{G_2G+GG_1}{GG_1}=\frac{G_2G_1}{GG_1}=\frac{M}{m}.$$

This shews that G_2G has to be divided externally at G_1 in the ratio $\frac{M}{m}$.

Hence, from G and G_2, set off parallel lines, in the same sense, representing the masses to scale. Join their end points and produce the line to cut G_2G in G_1, the M.C. required. This was the construction made in the example.

Mass-Centre of a Segment of a Circle. Draw a circular sector $OACB$ (Fig. 106) of radius 4·5″ and angle 150°. Construct the M.C. G_1 of this sector as before. Divide OM (Fig. 106) into three equal parts and take $OG_2 = \frac{2}{3}OM$. Through G_1 and G_2 draw parallels and set off $G_1K_1 = p$, the perpendicular from M on OA and G_2K_2 = the arc AC.* (In Fig. 106 $\frac{2}{3}$ of these distances are set off.) Join K_2K_1 cutting OC at G, the M.C. of the segment.

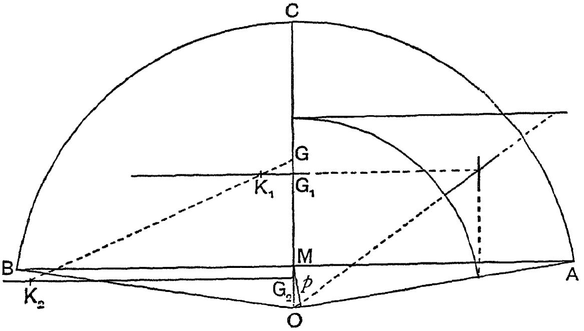

FIG. 106.

Proof. The problem is to find the M.C. of the sector area and the negative area of the triangle OAB.

$$\text{The area of the sector} = \tfrac{1}{2}\,\text{radius} \times \text{arc (p. 55)}$$
$$= OA \times \text{arc } AC.$$

$$\text{The area of the triangle} = OM \, . \, MA$$
$$= OA \, . \, p \, ; \left(\text{since } \frac{OA}{AM} = \frac{OM}{p}\right);$$

$$\therefore \; \frac{\text{area of sector}}{\text{area of triangle}} = \frac{\text{arc } AC}{p}.$$

Since the area of the triangle must be considered as negative, p and the arc AC must be set off in the same sense.

(66) Construct the M.C. of a circular area of radius 3 inches having a circular hole of radius 1″ cut out, the distance apart of the centres being 1·5″. (Areas are proportional to radii squared.)

---

* These distances must be set off very carefully, especially when the angle of the sector is very small or nearly 180°.

(67) Construct the M.C. of the area as above, the radii being R and r, and the distance apart of the centres being c. Construct the ratio of the squares as in Fig. 107.

$$\left(\frac{r}{R}=\frac{R}{x};\quad \therefore\ \frac{x}{r}=\frac{R^2}{r^2}\right).$$

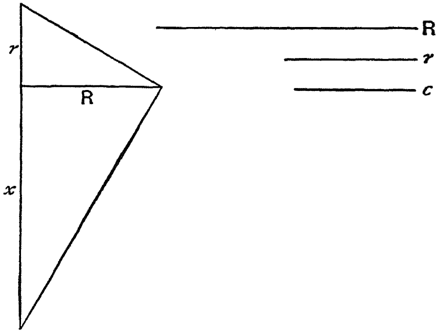

FIG. 107.

(68) Construct the M.C. of the figure shewn (Fig. 108).

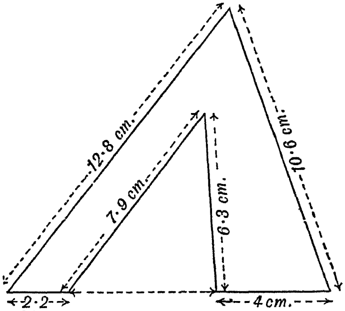

FIG. 108.

(69) Find the M.C. of a rectangle of sides 5·64″ and 7·85″ with a square of side 2·83″ cut out, the distance apart of their centres being 1·5″, and the line of centres being a diagonal of the rectangle. (Use a similar construction to that in Ex. 67 for getting lines proportional to areas.)

(70) Find the M.C. of a rectangle (5·64″ × 4·85″), having a circular hole of radius 1·4″ cut out, the distance apart of the centres being 1·5″.

(71) Draw a rectangle of sides 4 and 3 inches. On the 3″ side as diameter describe a semicircle inside the rectangle; now suppose it cut away. Find the M.C. of the remaining area.

Vectors in a Plane. **Parallel vectors are said to be LIKE vectors, they can all be expressed as multiples of any other parallel vector.**

Two non-parallel vectors are independent, *i.e.* one cannot be expressed in terms of the others.

Draw any two non-parallel vectors α and β; see that the sum or difference of any multiples of these is the third side of a triangle, which can only be zero when α and β have the same direction, or the two multiples are zero. If, therefore, we have an equation $a\alpha = b\beta$, where α and β are independent, it can only be satisfied when $a = b = 0$.

Similarly, if $$7\alpha + 3\beta = a\alpha + b\beta,$$

then $$(7 - a)\alpha = (b - 3)\beta,$$

and a must be 7 and b must be 3 if the vectors are not parallel.

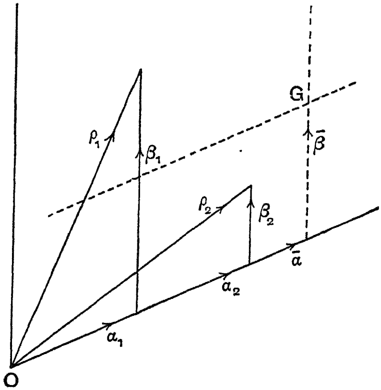

FIG. 109.

Scalar Equations for Mass-Centre. A number of masses $m_1, m_2, m_3, \ldots$ are at points whose position vectors are $\rho_1, \rho_2, \rho_3, \ldots$. Take any two axes through O (Fig. 109), and let the

components of ρ_1 parallel to the axes be α_1 and β_1,

" ρ_2 " " α_2 and β_2,

....

Then
$$\rho_1 = \alpha_1 + \beta_1,$$
$$\rho_2 = \alpha_2 + \beta_2,$$
$$\ldots\ldots\ldots\ldots\ldots$$

Let $\bar{\rho}$ be the position vector of the mass-centre and $\bar{\alpha}$ and $\bar{\beta}$ the components of $\bar{\rho}$.

Then, for the mass-centre,
$$\bar{\rho}\Sigma m_1 = \Sigma m_1 \rho_1 ;$$
$$\therefore \quad (\bar{\alpha} + \bar{\beta})\Sigma m_1 = (m_1\alpha_1 + m_1\beta_1 + m_2\alpha_2 + m_2\beta_2 + \ldots)$$
$$= \Sigma m_1\alpha_1 + \Sigma m_1\beta_1,$$
by rearrangement of the terms.

But $\bar{\alpha}$, α_1, α_2, ... are like vectors, and so are $\bar{\beta}$, β_1, β_2, ..., and α_1 and β_1 are independent.

Hence $\quad \bar{\alpha}\Sigma m_1 = \Sigma m_1\alpha_1$ and $\bar{\beta}\Sigma m_1 = \Sigma m_1\beta_1$.

If, then, x_1 is the length of α_1, y_1 of β_1, $\bar{x}$ of $\bar{\alpha}$, etc.,
$$\bar{x}\Sigma m_1 = \Sigma m_1 x_1 \text{ and } \bar{y}\Sigma m_1 = \Sigma m_1 y_1,$$
two scalar equations, each of which determines a line on which the mass-centre must lie, the point of intersection of the two lines being G the mass-centre.

Generally, it is convenient to take the axes perpendicular, and in this case $m_1 y_1$ is the mass at a point multiplied by the perpendicular distance of the point from the axis of x, and is called the **mass moment** about Ox.

$\Sigma m_1 y_1$ is then the sum of the mass moments about the axis of x.

The whole mass Σm_1, supposed concentrated at the mass centre, is called the **resultant mass.**

We have then the theorem :

The sum of the mass moments about any line is equal to the moment of the resultant mass.

Graphical Construction for M.C. The construction given on p. 94 for the position of the M.C. of points in a line can be applied to points in a plane, the construction being made for two intersecting lines. A better method, however, is the link polygon construction, given on p. 299.

EXAMPLE. *Find the* M.C. *of masses* 2, 3, 5, *and* 1 *lbs. at points whose coordinates are* (1, 1), (2, 3), (4, 1) *and* (3, 2).

Draw the axes of coordinates (Fig. 110), and mark the coordinates of the points on each axis; draw a line bisecting the angle between them, and set off along this line the masses to any convenient scale

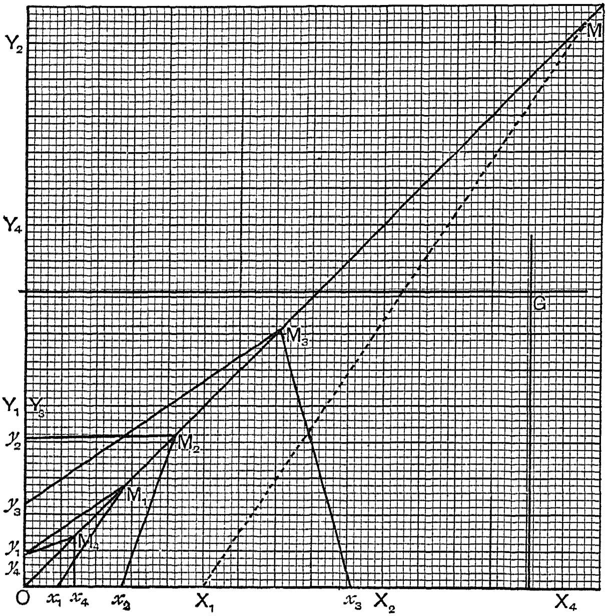

FIG. 110.

In Fig. 110, $OX_1 = 1$, $OY_1 = 1$ and $OM_1 = 2$ (on mass scale); $OX_2 = 2$, $OY_2 = 3$ and $OM_2 = 3$ .... ($OX_3 = 4$ is not shewn.) The suffixes indicate the order of the points in the example.

Join M ($OM = 11$, the sum of the masses) to 1, 2, 4, 3 on the x axis, and mark the points where parallels to these lines from 2, 3, 5, 1 on the mass axis respectively cut the x axis, x_1, x_2, x_3, x_4.

Then
$$\frac{OM_1}{Ox_1} = \frac{11}{1} \text{ or } 11x_1 = 2.1.$$

Similarly $\frac{OM_2}{Ox_2}=\frac{11}{2}$ or $11x_2=3\,.\,2,$

$$11x_3=5\,.\,4,$$
$$11x_4=1\,.\,2.$$

Hence $11(x_1+x_2+x_3+x_4)=2\,.\,1+3\,.\,2+5\,.\,4+1\,.\,2=\Sigma m_1x_1$.

Add by a straight edged strip x_1, x_2, x_3 and x_1, the sum is $\bar{x}$.

Make a similar construction on the y axis and obtain $\bar{y}$.

Mark the point whose coordinates are $\bar{x}$ and $\bar{y}$; it is the mass centre, G.

(72) Masses are given by lines of length 5·1, 2·3, 1·5 and 2·15 cms.; the coordinates in inches of the points are (0, 1·1), (2·3, 0), (3·2, 1·7), (2·2, 4·3). Find the mass-centre by construction, and test by calculation from the formula $\bar{x}=\frac{\Sigma m_1x_1}{\Sigma m_1}$, $\bar{y}=\frac{\Sigma m_1y_1}{\Sigma m_1}$.

Graphical Construction for the M.C. of any area, irregular or otherwise. Transfer the figure given in Fig. 111 to your drawing paper, and draw its axis of symmetry XY. Divide it up into strips parallel to the base, and draw the first equivalent figure as in the construction on p. 62, only take the point O in the base and a equal to the height of the figure. Divide the height at G so that

$$\frac{XG}{XY}=\frac{A_1}{A},$$

where A and A_1 are lines proportional to the areas of the given and the equivalent figure. G is the M.C.

* **Proof.** This is very similar to the proof on p. 63.

In Fig. 111, AB is one of the very thin strips, parallel to the base, into which the area is supposed to be divided. The mass of each of these strips may be supposed concentrated at the mid-point, *i.e.* in the axis of symmetry, XY.

If m is the mass of any one of these strips and y is the distance from the base, then, if $\bar{y}$ is the distance of the M.C. from the base

$$\bar{y}\Sigma m=\Sigma my.$$

But m is proportional to the area of the strip, hence if $AB=x$ and h is the thickness of the strip, m is proportional to hx, and therefore

$$\bar{y}\Sigma hx=\Sigma hxy.$$

Now (see Fig. 57) $\frac{x}{a}=\frac{x_1}{y}$, hence $xy=ax$,

and $\bar{y}\Sigma hx=\Sigma hax_1$;

$\therefore\ \bar{y}$. area of given Fig. $=a\Sigma hx$

$= a$. area of first equivalent figure,

and $\bar{y}=a\frac{A_1}{A}$.

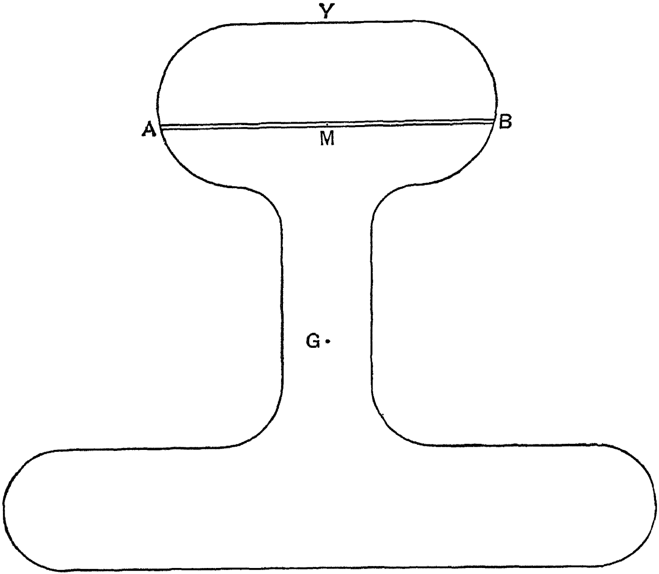

FIG. 111.

If the area has not an axis of symmetry, $\bar{y}$ only determines the distance of the M.C. from XX. The process must therefore be repeated for another line which intersects XX (preferably at right angles to it), and the distance of the M.C. from this line must be determined. From these two distances the M.C. can be determined as the intersection of two lines.

Another method is given on p. 299 in connection with the Link Polygon.

MISCELLANEOUS EXAMPLES. III.

1. Draw a square *ABCD* of side 3 inches to represent a square field of side 300 yards. A man starting at *A* walks round at 80 yards a minute, another man starting at *D* at the same instant, and walking at 100 yards a minute, begins to overtake him. Construct the relative displacements at the end of the 1st, 2nd, 3rd, 4th, 6th, 9th, 12th and 15th minutes.

2. Construct the minimum relative displacements when the men are on adjacent sides.

3. A toy gun is pointed at an elevation of 45°; on firing it begins to recoil with a speed of 5 ft. per sec. (horizontally), the speed of the shot relative to the gun is 30 ft. per sec., construct the true velocity of the shot.

4. A sailing boat is going N.W. at 8 miles an hour, a sailor moves across the deck from the S.W. at 2·7 miles an hour, a current is flowing at 3·6 miles an hour 15° S. of E.; what is the velocity of the sailor relative to the current?

5. Find the centre of mean position of five points whose coordinates in cms. are (2·1, 3·3), (4·7, 1·8), (2·6, 1·75), (1·95, 4·6), (0·75, 6·25).

6. Find the mass-centre of the above points supposing masses of 3, 4·1, 2·8, 7·3 and 4·6 grammes to be concentrated at the points, first by the vector polygon method, secondly by the graphical construction of p. 113, and finally by calculation.

7. Find the M.C. of the part of a circular area between two parallel lines at distances 3·74 and 2·66 inches from the centre, the radius of the circle being 4·5 inches. First treat it as the difference between two segments and then by the strip division method.

8. Find the M.C. of a circular arc and its chord, the arc subtending an angle of 135° at the centre of a circle of radius 3·7″.

9. Masses of 3, 8, 7, 6, 2, 4 grammes are placed at the vertices *A*, *B*, ... of a regular hexagon; construct the position of the M.C.

10. Draw a circular arc of radius 3″ and one of radius 2″, the distances apart of the centres being 2″. Find the M.C. of the lens shaped area between the two arcs by the method of strip division.

11. A horizontal wooden cylinder rests on the top of a rectangular block of wood, the radius of the cylinder = width of block = 2·52 ft., the height of the block is 7·86 ft., and the length of the cylinder and depth of block are the same. Find the M.C. (Treat as a circle on a rectangle.)

12. A steamer which is steaming in still water due S.E. at a speed of 14 knots enters a current flowing due W. at a speed of 2 knots. Determine in any way you please the actual velocity of the steamer when in the current and the direction in which she will travel.

If it is desired to maintain a due S.E. course and to cover exactly as much distance per hour in this direction as when in the still water, what course would the steamer require to steer, and what must be the speed of the ship in regard to still water? (Military Entrance, 1905.)

13. A uniform iron girder has a cross-section of the given form (Fig, 112). Determine the position of the centre of gravity of the section.

(Naval Cadets, 1904.)

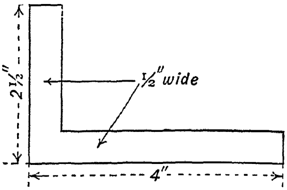

FIG. 112.

14. Fig. 113 represents a figure formed of a rectangle and an isosceles triangle. Find and mark the position of its centre of gravity.

(Naval and Engineer Cadets, March, 1904.)

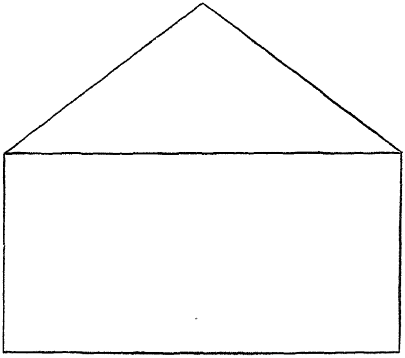

FIG. 113.

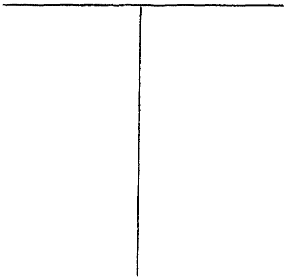

FIG. 114.

15. The letter T in the diagram (Fig. 114) is made of wire of uniform thickness. Find its centre of gravity (M.C.), stating your method.

(Naval Cadets, 1903.)

16. Fig. 115 represents a hexagon frame, the length of each side being 2 inches; equal masses of 4 pounds each are placed at the four corners *a*, *b*, *d*, *e*, and a mass of 8 pounds is placed at the corner *c*. The mass of each of the six sides of the hexagon frame is $1\frac{1}{2}$ pounds. Find the common centre of gravity (M.C.) of the whole system.

(Military Entrance, 1905.)

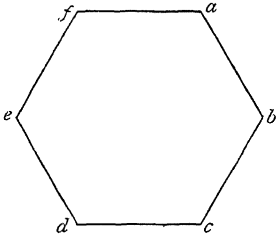

FIG. 115.

17. A cistern, without lid, whose thickness may be neglected, measures 3 ft. 6 in. in height by 2 ft. 3 in. by 3 ft. 3 in. Find the position of its centre of gravity.

18. *A* and *B* are two points 20 miles apart. At noon one man starts from *A* to walk to *B* at the rate of 4 miles an hour, and at 2 p.m. another man starts after him on a bicycle at 10 miles an hour. Draw a diagram on your ruled paper to show how far they are apart at any given time, and at what times they pass any given point between *A* and *B*. [Scale to be 5 mile = 1 inch, and 1 hour = 1 inch.]

Also, find from your diagram or otherwise when and where the cyclist overtakes the man walking. (Engineer Students, Navy, 1903.)

19. Two small spheres, of weights 4 ounces and 7 ounces, are placed so that their centres are 5 inches apart. How far is their centre of gravity from the centre of each? (Engineer Students, 1903.)

20. Given the centre of gravity of a body, and that of one of its parts, explain how to find the centre of gravity of the remaining part.

ABDC is a rectangular lamina of uniform density; *E* is the middle point of *AB*; join *DE*; find the perpendicular distances of the centre of gravity of *BCDE* from the sides *BC* and *CD*. (B. of E., II.)

21. A river which is 2 miles wide is flowing between parallel straight banks at the rate of 4 miles an hour. A steamer starts from a point *A* on one bank and steers a straight course at 7 miles an hour. Show on a graph the distance above or below *A* of her point of arrival at the other bank, as a function of the inclination of her course to the direction of the river. (Naval Cadets, 1905.)

22. Explain the method of determining the motion of one body relative to another. To a passenger on a steamer going N. at twelve miles per hr., the clouds appear to travel from the E., at 8 miles per hr.; find their true velocity.

Two steamers are at a given instant 10 miles apart in an E. and W. line; they are going towards each other, one N.E. at 20 miles per hr., and the other N.W. at 16 miles per hr. Find how near they approach.
(Inter. Sci., 1901.)

23. Find the centre of gravity of a triangular frame formed of three uniform bars of equal weight. Where must a mass equal to that of a uniform triangular plate be fixed on the plate so that the mass centre of the whole may be at the middle of the line joining a vertex to the point of bisection of the opposite side. (Inter. Sci., 1901.)

24. Explain the phrase "velocity of one body relatively to another body."

25. Two level roads are inclined at an angle of 60°. Two motors, each half-a-mile from the junction are being driven towards it at speeds 10 and 11 miles an hour. Find the velocity of the first motor relative to the second, and the distance the motors are apart after 2 minutes 50 seconds.
(B. of E., II., 1904.)

26. A ship A that steams 23 knots sights another ship B to the N. at a distance of 1·4 sea miles, and steaming E. at 19 knots. In what direction must A steam in order that her motion relative to B may be directly towards B? and in what time does she reach B? A knot is a speed of 1 sea mile per hour.

Shew that if A does not know B's speed she can deduce it from her own by steering in such a direction as to keep due S. of B. If this direction is 78° E. of N., and the other data are as already given, what is B's speed? (Military Entrance, 1906.)

27. Find the position of the centre of gravity of a circular sector. Find the distance of the centre of gravity of a circular segment from its chord. (B. of E., II., 1906.)

28. Ox and Oy are two lines at right angles; P and Q are points moving from O to x and from O to y, with speeds 7 and 12 respectively. At the same instant $OP=8$ and $OQ=5$. Find the velocity (speed) with which they are separating from each other, and explain whether or not the velocity of separation is their relative velocity. (B. of E., II., 1903.)

29. Give an instance of a moving body that is at rest relatively to another moving body. State how the relative velocity of one point with respect to another point can be found.

Two points A and B are moving with equal speeds and opposite senses round a given circle; at the instant that the arc between them is a quadrant, find the relative velocity of A with respect to B. (B. of E., II., 1905.)

CHAPTER IV.

CONCURRENT FORCES.

EXPERIMENTS.

Expt. I. Lay an envelope or sheet of paper on a fairly smooth table. Push it by a pencil parallel to the shorter edge (i) near one corner, (ii) near the middle. Notice that the motion is quite different in the two cases, even though the push is otherwise the same. Does the effect of a force on a body depend on its line of action (axis)? Think of other simple experiments illustrating this point.

For the remaining experiments the following apparatus is necessary: A vertically fixed drawing board and paper; light freely-running pulleys which can be clamped round the board in any desired positions; a set of weights from 5 to 1000 grammes; one or two scale pans of known weight; some light, tough, stiff cardboard; strong, black, fine thread; some small polished steel rings, about the size of a threepenny piece; some thin, strong wire (for making little hooks); and a spring balance.

Expt. II. Fasten, by means of loops, two threads to one of the steel rings, and attach 100 gramme-weights to the other ends. Put the threads over two pulleys as indicated in Fig. 116, and let the whole come to rest. Take another piece of thread in the hands, and, stretching tightly, see if the two threads are in a straight line.

The function of the pulleys is only to change the directions of the forces due to the weights; any effect due to their having friction may be minimised by good lubrication.

What are the forces acting on the ring, neglecting its weight, in magnitude, direction and sense? Draw lines representing these in magnitude, direction and sense; these are the vectors of the forces. Add these vectors. What is their sum? Why may the weight of the ring be left out of consideration? See if equilibrium is possible with different weights W_1 and W_2.

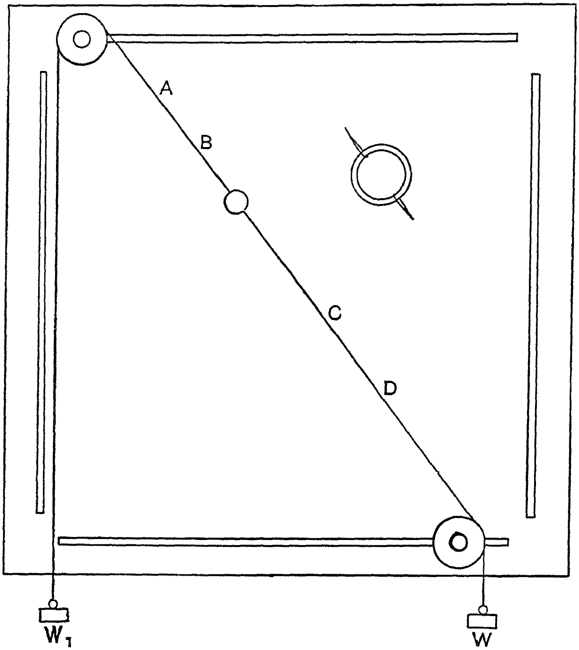

FIG. 116.

What is the pull at B on the part BA? What is the pull at B on the part between B and the ring? In what respect do these pulls differ?

EXPT. III. Replace the ring by a piece of cardboard and attach the threads by wire hooks passing through two holes punched in the card. See that the sum of the vectors of the forces is again zero. Mark on the card, points A, B, C, in a line with the threads. Punch holes at these points, and insert the lower hook in turn through each of these holes. Is equilibrium still maintained? Does it matter at what point in its axis a force may be supposed applied to a rigid body?

(1) Four hooks A, B, C, D are connected together by three strings AB, BC, CD, Fig. 117. Weights $W_1 = 100$ grammes, $W_2 = 50$ grammes, $W_3 = 150$ grammes are attached by long strings to B, C and D, and the whole is

suspended from the hook A fixed to a beam or wall. If M_1, M_2 and M_3 are points in AB, BC and CD, what is the pull at M_3 on C, M_2 on B, and at M_1 on A?

Verify at M_2 by inserting a spring balance there.

EXPT. IV. Attach three threads by loops to one of the rings and suspend weights 200, 150, and 100 grammes as indicated (Fig. 118). Let the ring take up its position of equilibrium. Mark two points on the drawing paper under each thread. A set square does very well for this purpose, if placed approximately perpendicular to the plane of the board, but a small cube or right prism is better. Indicate the sense of each pull on the ring. Remove the drawing paper and draw to scale the vectors, α, β and γ, of the forces acting on the ring, and find their sum.

Are the lines of action of the forces concurrent?

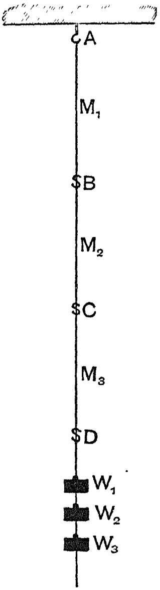

FIG. 117.

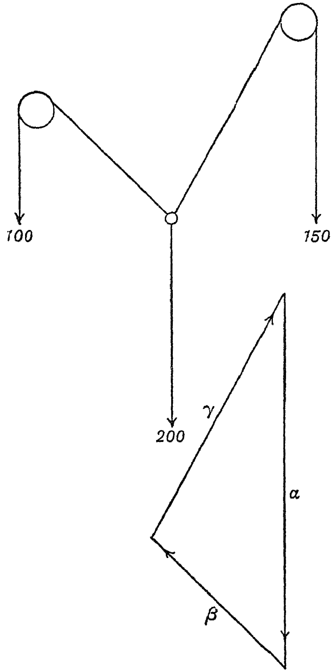

FIG. 118.

Repeat the experiment and drawing for other weights.

Is equilibrium possible with 50, 60 and 150 grammes?

EXPT. V. Use three pulleys and four threads with attached weights of 80, 120, 200 and 120 grammes. Mark the axes and the senses of the pulls on the ring as before, and find the sum of the vectors.

Perform a similar experiment and construction using five different weights. Notice in each case whether the forces are concurrent or not.

EXPT. VI. Draw a triangle *ABC* (Fig. 119) on cardboard having sides 3·4, 3·6 and 5·2 inches, and give the boundary a clockwise sense. Draw concurrent lines parallel to these sides, indicating the senses as in Fig. 119. Punch holes on these three lines and cut away the card as indicated by dotted lines. Fix the card, with one axis vertically downwards, on the drawing board by pins. Adjust the threads, hooked through the holes, so as to lie over the lines and attach weights proportional to the corresponding sides of the vector triangle. Remove the pins and notice if the card remains in position.

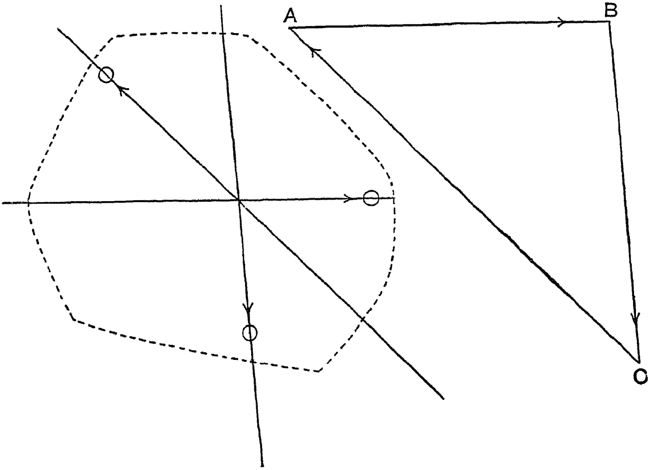

FIG. 119.

Perform a similar experiment starting (i) with a quadrilateral of sides 5, 3, 2 and 6 inches, the senses being the same way round, (ii) with a pentagon; the axes must be concurrent in both cases.

EXPT. VII. Attach three weights to a card (one thread hanging vertically and two passing over pulleys). The card will take up some position of equilibrium. See if the axes of the forces are concurrent. Attach four weights to the card, and after the card has taken up its position of equilibrium see if the axes are concurrent.

Perform a similar experiment with five weights.

EXAMPLE. *A student repeating Expt. IV. has a vertical pull of* 40 *grms. weight on the ring, the two parts of the left-hand thread supporting* 25 *grms. weight make an angle of* 45° *with one another. What was the third weight used, and what was the angle between the two parts of its thread?*

Set off $OA = 4''$ (Fig. 120) vertically downwards, $OB = 2{\cdot}5''$ at an angle of 45° with OA. Join AB. Then OAB is the vector triangle of the forces, and $AB = 2{\cdot}85''$ gives the third pull on the ring of magnitude 28·5 grms. weight (nearly). The direction is given by the angle OAB and the sense is from A to B.

Measure OAB (*a*) with a protractor, (*b*) by a scale of chords, (*c*) by means of the tangent of the angle (*i.e.* measure p on the 2-inch scale, p being the perpendicular to AB drawn at $2''$ from A, and find the angle from the table of tangents). See that these three results are approximately 38·7° each.

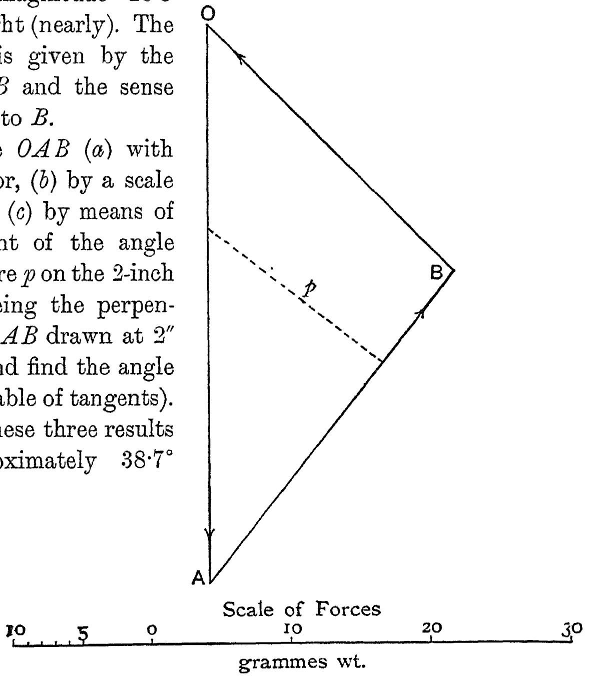

FIG. 120.

EXAMPLE. *Another student used weights* 20, 30 *and* 32 *grms., the last giving the vertical pull. What were the angles between the threads attached to the ring?*

Set off AB (Fig. 121) vertically downwards $=3{\cdot}2''$; draw circles with A and B as centres and of radii 2 and 3 inches to intersect in C. Draw concurrent lines P_1O, P_2O, WO parallel to the sides of this triangle.

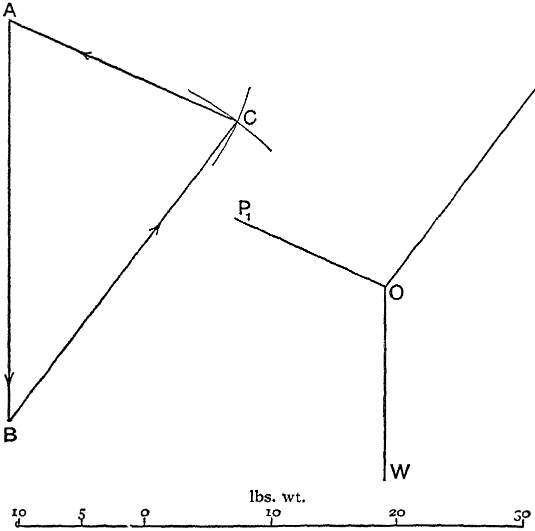

FIG. 121.

Evidently the angles between the threads are the supplements of the angles of the triangle; measure the angles of ABC by the scale of chords. Approximately they are 37·5°, 65·8° and 76·7°.

Notice that since the construction may be done in two ways, viz. the 2″ circle from either A or B, it is impossible to say which weight was put on the left-hand pulley.

In all solutions of statical problems by graphical methods it is necessary to complete the solution either by drawing out the force scale or by giving it in cms. or ins.

EXAMPLE. *A weight of* 20·6 *tons is suspended by ropes of length* 7 *and* 8 *ft. from two hooks in a horizontal line distant apart* 5 *ft. Find the pulls of the ropes on the weight* (*the tensions in the ropes*).

First draw a figure *ABC* (Fig. 122) representing the position of the ropes to scale (1″ to 1′).

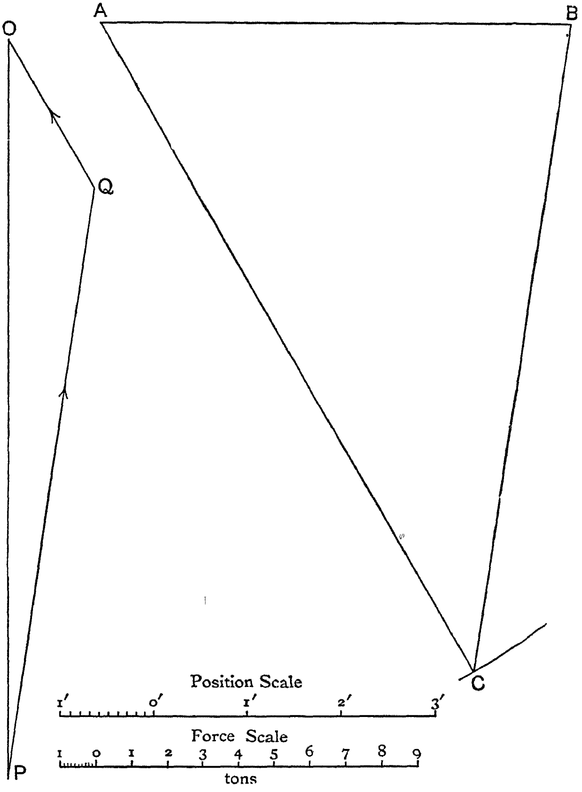
O
A
B
Q
C
P
Position Scale
1′
0′
1′
2′
3′
Force Scale
1
0
1
2
3
4
5
6
7
8
9
tons

FIG. 122.

Then set off *OP* downwards to represent 20·6 tons weight (1 cm. to 1 ton). From the ends of *OP* draw *OQ* and *PQ*, parallel to *AC* and *BC*, and measure *PQ* and *QO* on the ton scale.

EXAMPLE. *From a telegraph pole radiate five lines (in the same horizontal plane). The pulls of four of them on the pole are known, find the pull of the fifth; given a pull due E. of* 30 *lbs. weight, one due S. of* 40, *one S.W. of* 25, *one N.W. of* 41·4.

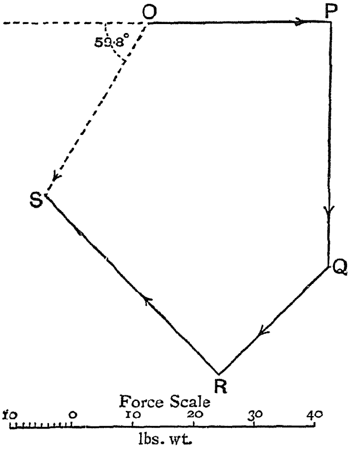

FIG. 123.

Set off OP (Fig. 123) horizontally to the left $=3''$, PQ vertically downwards $=4''$, $QR=2·5''$ so that $PQR=135°$, QR perpendicular to $RS=4·14''$, then **SO** gives ($3·2''$ in length) the fifth pull in magnitude, direction and sense.

(2) In a tug-of-war A, B, C, D are opposed to A_1, B_1, C_1, D_1, D and D_1 being the end men. The pulls of A, B, C, D are given by lines of lengths 2·8, 3·4, 3·5 and 3·9 cms. respectively, those of A_1, B_1 and C_1 by 2·45, 3·25 and 3·75; scale 1″ to 100 lbs. wt. What is the least pull that D_1 must exert in order that his side may not be beaten? If D_1 exerts this force, give the tensions of the rope at points intermediate between the men.

(3) Three strings are fastened to a ring as in Expt. IV., two pass over smooth pulleys and bear weights P and Q, the third string hangs vertically and supports a weight R. If θ and ϕ denote the angles between the vertical and the threads attached to P and Q, find

(i) θ and ϕ when $P=Q=5$ lbs. wt. and $R=8$ lbs. wt.;
(ii) R and ϕ when $P=Q=5$ and $\theta=30°$;
(iii) θ and ϕ when $P=4$, $Q=5$ and $R=7$;

(iv) Q and ϕ when $P=7$, $R=9$ and $\theta=60°$;
(v) P and ϕ when $Q=12$, $R=11$ and $\theta=55°$;
(vi) P and Q when $R=7$, $\theta=35°$ and $\phi=50°$;
(vii) θ and Q when $R=13$, $P=8$ and $\phi=35°$;
(viii) θ and P when $R=11$, $Q=6$ and $\phi=30°$;
(ix) P and Q when $R=14$, $\theta=40°$ and Q is perp. to P;
(x) R and Q when $\theta=35$, $\phi=70°$ and $P=5{\cdot}5$. (Notice that the angle between P and Q in the vector triangle is 75°.)

(4) Three threads fastened to a ring bear weights of 35, 27 and 25 grammes as in Expt. IV.; draw the vector triangle of the forces, and by measurement determine the angles between the threads.

(5) Three threads are fastened to a ring, as in Expt. IV., the vertical load is 135 grammes weight, the acute angles the sloping strings make with the vertical are 25° and 50°; find the other two weights.

(6) A load of 27 lbs. is supported by two strings attached to hooks; if the strings make angles of

(i) 27° and 48°; (ii) 40° and 70°; (iii) 50° and 50°;

with the vertical, determine the pulls on the hooks.

(7) To two hooks A and B are fastened ropes which support a load of 3 cwts. The distance apart of A and B is 7 ft. and A is 3 ft. higher than B; if the lengths of the ropes attached to A and B are 5 ft. and 4 ft., what are the tensions in the ropes, *i.e.* what are the pulls of the ropes on the load and on the hooks?

(8) A weight of 12,000 grammes is supported by strings from two hooks A and B in the same horizontal line. The distance apart of A and B is 15 ft. and the string attached to B is 12·2 ft. Find the pulls on the hooks when the length of the string attached to A is

(i) 15; (ii) 14·4; (iii) 10; (iv) 5; (v) 4; (vi) 3 ft.

(9) Draw a graph shewing the relation between the pull on the hook in Ex. 8, and the length of the string attached to it, as the string varies in length from 3 to 15 ft.

Take two axes on squared paper and an origin. From the diagrams of position and of vectors, the pull corresponding to the string length 3, 4, 5, 10 and 14·4 ft. can be found. Plot points having as abscissae the lengths of the string and as ordinates the corresponding pulls. Join the points by a smooth curve. From the graph read off the pulls corresponding to string lengths of 12 ft. and 7 ft., and the lengths corresponding to pulls of 5000 and 17,000 grammes weight.

If the string could stand a pull of 12000 grammes only, what would be the least length of string that could be used?

(10) Draw a graph shewing the relation between the length of the variable string and the pull on the other hook.

(11) Draw a graph shewing the relation between the two pulls on the hooks,

(12) In repeating Expt. V. a student used five weights. The directions and magnitudes of four of the pulls being as given in Fig. 124, what was the magnitude of the fifth weight, and in what direction and sense was the pull exerted by it?

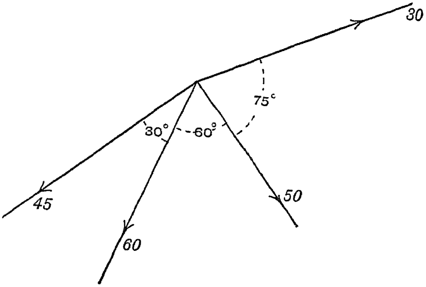

FIG. 124.

(13) Four concurrent forces are in equilibrium and act in the lines indicated (Fig. 125). If $P=18$ and $Q=25$ lbs. weight, find R and S in magnitude and sense. (Find the vector giving the sum of the two known vectors, from its end points draw parallels to R and S. This can be done in two ways; but the vectors parallel to R and S are the same in each case. Since the forces are concurrent and the vector polygon is closed, these vectors must give the forces in magnitude, direction and sense.)

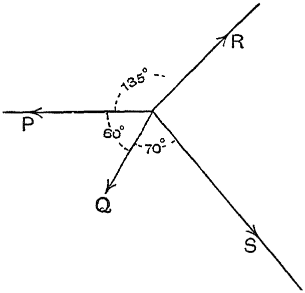

FIG. 125

(14) A wheel has six central equi-spaced spokes, in four consecutive spokes the pushes on the axis are 0·32, 0·72, 1·15 and 0·84 lbs. wt.; what are the actions of the remaining two spokes on the axis?

(15) A weight of 10 lbs. hangs vertically by a string from a hook. The weight is pulled horizontally so that the string makes an angle of 37° with the vertical. What is the magnitude of the horizontal pull and what is the pull of the string on the weight?

(16) Draw a graph shewing the relation between the pull S on the hook and the horizontal pull H in Ex. (15) as H increases gradually from 0 to 10 W.

For any given value of H the vectors form a triangle OAB (say), OA representing the weight W, AB the pull P and BO the pull of the string on W. At B draw BP perpendicular to AB and of length BO. Go through this construction when AB represents 1, 2, ... 10 lbs. weight, and join the points P so obtained by a smooth curve. This curve is the one required, for A being the origin of coordinates and AB and AO the axes, the coordinates of P are the values of H and S necessary to give equilibrium.

(17) In Fig. 126 AB is a light rod with a weight of 11 lbs. at B; the rod can turn freely round A. B is pushed perpendicularly to AB with a force of 4 lbs. weight. Find the position of AB and the pull on A.

[Since the angle at B is a right angle, describe a semicircle on the vector representing 11 lbs. weight, and set off in this a line representing a force of 4 lbs. weight; the closing line of the triangle gives the direction of AB and the force it exerts on B].

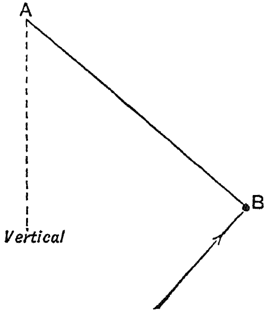

FIG. 126.

(18) Draw a graph shewing the relation between the push at B (P) and the pull S on A in Ex. (17) as P increases from zero to 11 lbs. weight.

(19) In Fig. 127 A is a fixed hook and C a smooth pulley, B a smooth ring to which the threads AB, BC and BW are attached. If $B\hat{C}D=30°$, $A\hat{B}C=85°$, find the pull on A and the weight W.

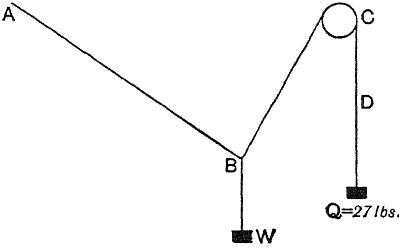

FIG. 127.

The angles remaining constant, draw a graph shewing the relation between Q and W.

(20) Two cords are fastened to a ring at C, and, hanging over pulleys at A and B, bear weights of 12 and 17 lbs. Find the force in magnitude, direction and sense, with which C must be pulled in order that, with AC and BC making angles of 60° and 80° with the vertical, there may be equilibrium.

(21) A load of 5 cwts. is suspended from a crane by a chain of length 20 ft.; a doorway is opposite the load and 5 ft. distant; with what force must the load be pulled horizontally to cause it just to enter the doorway?

(22) Strings of length 5 and 3·2 ft. respectively are fastened to a floor at points distant 4·3 ft. apart; the other ends are attached to a smooth ring which is pulled by means of a string making 30° with the vertical with a force of 50 lbs. weight. The three strings being in one plane, find the other pulls on the ring.

(23) OA and OB (Fig. 128) are the axes of two forces, a is the vector of the force in OA, c is the *magnitude* of a third force which, acting through O, is in equilibrium with the other two forces. Find the vectors of the other forces. How many solutions has the problem? Can you choose a magnitude for c, so that there shall be only one solution? Can you choose a magnitude for c so that equilibrium is impossible? What is the least magnitude of c consistent with equilibrium?

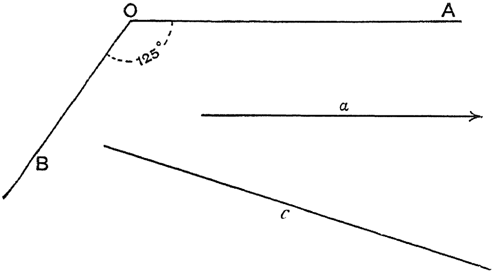

FIG. 128.

(24) A weight of 50 lbs. is supported from A and B as in Fig. 129.

(i) Find the pulls on A and B.

(ii) If a man pulls in the direction and sense OC with a force of 10 lbs. weight, find the alteration in the pulls on A and B.

(iii) If he pulls in the opposite sense, find the alteration in the pulls on A and B.

(iv) Find the pull so that there may be no tension in OB.

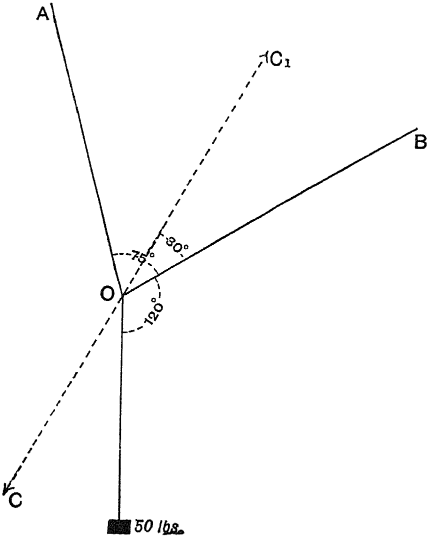

FIG. 129.

The Foundation of Statics. Such experiments as have been performed cannot be considered in themselves as the best foundation for the science of mechanics, the true basis for which must be sought in far more generalised experience. Such generalised experience was summed up by Newton (p. 135). Deductions made from such experiments as those detailed must consequently be regarded as tentative only. The experiments, however, have the great advantage of giving a reality to notions concerning the action of forces which descriptive matter fails to impart.

Deductions from Experiments. The experiments now performed all relate to the action of forces on rigid bodies. Force without some body (mass) acted on is a meaningless term; forces do not act on points but on masses, and such an expression as "forces acting at a point" means only that the lines of action of the forces are concurrent.

Expt. I. shewed that a force is determined only when we know some point in its line of action in addition to its magnitude, direction and sense

Expts. II. and III. shewed (i) that a body under the action of two forces is in equilibrium when, and only when, the forces differ in sense alone; (ii) a force acting on a rigid body may be supposed to act anywhere in its axis.

By a **rigid body** is meant one which retains the same relative position of its parts under the action of all forces. Any body which maintains its shape unaltered, or for which the change is too small to be measurable under the action of certain forces, may be considered as rigid for *those* forces. The paper in Expt. I. was practically rigid for the forces acting on it; it is, however, quite easy to apply forces to it that would change its shape. If a set of forces deform a body, but after a time the body takes up a new shape which does not alter while the forces are unchanged, such a body after deformation may be treated as rigid for those forces. For non-rigid bodies we must know not only the axis of the force, but also its point of application.

Expts. IV. to VI. shewed that if a rigid body, acted on by concurrent forces, is in equilibrium, the sum of the vectors of the forces is zero; and conversely, when the sum of the vectors of the forces is zero and the axes concurrent, the body is in equilibrium.

Expt. VII. shewed that when a body is in equilibrium under the action of three forces, their axes are concurrent; but that in general for four or more forces the axes are not concurrent when there is equilibrium.

Rotors. Any quantity which, like a force, requires for its specification the magnitude, direction, sense and a point on its axis, is called a **rotor quantity** (Clifford).

Such quantities may be represented geometrically by rotors, *i.e.* vectors localised in definite straight lines. The rotor may be specified by giving its vector and a point on its line of action. It is, however, usual and convenient to give

(i) the axis, (ii) the vector,

so that the direction is given twice over.

To avoid confusion in graphical work, the axes of the forces (rotors) should be drawn on a different part of the paper from the vectors giving the magnitudes, direction and senses of the forces.

Equilibrant. When a body is in equilibrium under the action of a number of forces, the forces themselves are, for shortness, often spoken of as being in equilibrium. For such a system of forces any one may be said to be in equilibrium with the rest, and from this point of view is called the **equilibrant** of the others.

Resultant. The equilibrant of such a system of forces would be in equilibrium with a certain single force differing from it only in sense (Expt. II.), and this reversed equilibrant would have the same effect, so far as motion is concerned, as all the rest of the forces together. The equilibrant reversed in sense is called the **resultant of the forces.**

It should be noticed that it has not been shewn that any system of forces has a resultant, but simply that if a system of forces is in equilibrium any one of them reversed in sense is the resultant of the rest, and would produce the same effect as regards motion as all the rest together.

Resultant of Concurrent Forces. To find the resultant of a number of concurrent forces acting on a body, add their vectors to a resultant vector and through the point of concurrence draw the axis of the resultant force parallel to its vector.

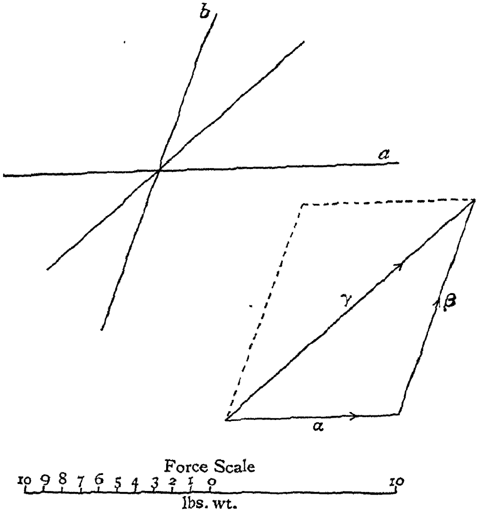

FIG. 130.

EXAMPLE 1. *Find the resultant of two forces of magnitude* 9·2 *and* 12·1 *lbs. weight acting towards the E. and towards a point* 66·5° *N. of E.*

Draw the axes a and b (Fig. 130) and add the vectors α and β of the forces (scale 1″ to 5 lb.), $\alpha+\beta=\gamma$, then γ is the vector of the resultant force. Through the point of intersection of a and b draw the axis c of the resultant.

Notice that if γ be set off along c and α and β along a and b, they form two adjacent sides and the concurrent diagonal of a parallelogram. That the magnitude, direction and sense of the resultant of two intersecting forces can be found, by adding the forces as vectors, is often, but badly, expressed by saying that forces are combined by the **parallelogram law.**

EXAMPLE 2. *Find the resultant of four forces of magnitudes* 13, 11, 9, 7 *kilogrammes weight whose axes are the lines joining a point* O *to points* A, B, C, D, *the five points being the vertices in order of a regular pentagon, and the senses being from* O *to* A, O *to* B, O *to* C *and* D *to* O.

Draw a circle of radius 2″, divide the circumference into five equal parts with dividers by the method of trial. Mark the five points in order O, A, B, C, D, then draw the vector polygon, α of length 13 cms. parallel to OA, β of length 11 cms. parallel to OB, γ of length 9 cms. parallel to OC, and, finally, δ parallel to OD, but having a sense from D to O. The vector σ joining the beginning of α to the end of δ is the resultant force in magnitude, direction and sense. Finally, draw a line through O parallel to σ, this is the axis of the resultant force.

(25) Find the resultant of two forces of magnitudes 16 and 18 kilogrammes weight, if they are directed N. and 75° E. of N.

(26) Three concurrent forces have magnitudes 23, 18, 15 lbs. weight, find their resultant in magnitude, direction and sense when the angles between them are 120° and 100°, and the forces all act outwards.

(27) If two forces are equal, shew that the resultant must bisect the angle between them.

(28) If the magnitudes only of two forces are given, in what relative directions should they act so that the resultant is (i) as big, (ii) as small as possible.

(29) α (Fig. 131) acts in the axis Ox, another force in Oy; find graphically the magnitude and sense of this force so that the resultant may be as small as possible.

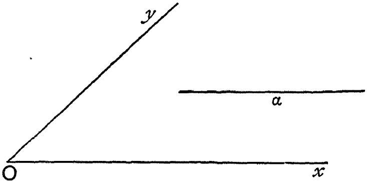

FIG. 131.

(30) Concurrent forces of magnitudes 12, 17, 10 and 8 lbs. weight are directed towards N., N.E., S.E. and 30° W. of S. respectively; find the resultant in magnitude, direction and sense.

(31) A wheel has six equi-spaced radial spokes; four consecutive spokes are in tension and pull on the hub with forces of 10, 15, 12 and 7 lbs. weight; find the resultant pull on the hub due to these spokes.

(32) A string *ABC* is fastened to a hook at *A*, passes round a free running pulley at *B* (*AB* horizontal), and is pulled in the direction *BC* where $ABC = 105°$ with a force equal to the weight of 17 lbs. Find the resultant force on the pulley at *B*.

(33) *ABCD* is a square of side 2″, a force of 11 lbs. weight acts from *A* to *B*, one of 7 lbs. from *D* to *A* and one of 3 lbs. from *C* to *A*; find the resultant force.

FORCE, MASS AND ACCELERATION.

Newton's Laws of Motion. The laws for the combination of concurrent forces deduced from Expts. I. to VII. are immediate deductions from Newton's famous Second Law of Motion. Stated shortly in modern language the law is—**a force acting on a particle (or body, if the axis passes through the M.C.), is measured by the product of the mass of the body and the acceleration produced.**

Acceleration being a vector quantity, force is a vector quantity, and since the force must act on the mass moved, it is a localised vector quantity or rotor.

The effect of two or more concurrent forces is found, therefore, by adding the corresponding accelerations as vectors. The single force, which would produce this resultant acceleration, is called the resultant force, and is measured by the product of the mass and this acceleration. To find, then, the resultant of a number of concurrent forces, add the forces as vectors; the sum gives the vector of the resultant force, and the axis of the force passes through the given point of concurrence.

A mass being in equilibrium when it has no acceleration, we see this will be the case, when, the axes being concurrent, the vector sum of the forces is zero, and conversely.

The equation connecting the three quantities, mass, force and acceleration is

$$\textbf{Force} = \textbf{Mass} \times \textbf{Acceleration.}$$

* **Mass and Weight.** If a foot and a second are units of length and time, a foot per second is the unit of speed, and a speed of a foot per second added per second (or a ft. per sec. per sec.) is the unit of speed acceleration. Further, if the unit of mass be a lb. mass, the unit of force must be that force which would give a lb. mass a speed acceleration of a ft. per sec. per sec.; or which would increase its speed every second by a ft. per sec. This follows at once from the equation: if the mass = 1 and the acceleration = 1, then the force must = 1.

We know that a body falling freely has a speed acceleration of 32·2 ft. per sec. per sec; hence if the mass be a lb. mass, the force acting on it is the lb. weight and is given by the equation,

$$\text{lb.-wt.} = \text{force} = 1 \times 32{\cdot}2.$$

In this system, then, the force on a falling lb. mass would be $1 \times 32{\cdot}2$ units of force; this is the weight of a lb. mass in these units. For statical purposes it is better, however, not to use this system, but to take the weight of the lb. mass as the unit of force.

The expressions lb. weight, force of a lb. weight, and lb. mass will often be met with; the first denotes the force with which the earth attracts the lb. mass; the second a force equal in magnitude to the weight of a lb. mass, but usually having a different direction.

In the C.G.S. system, similar double terms occur. The unit of mass is here a gramme, and the unit of length and time a centimetre and a second.

Unit force is then = gramme × an acceleration of a centimetre per sec. per sec., and is called a **dyne.**

The acceleration due to gravity in centimetres per sec. per sec. is 981, and, therefore, the weight of a gramme mass is 981 dynes. In statics, however, it is usual to consider the gramme weight as the unit of force, and thus we meet with the terms, gramme weight, force of a gramme weight, and gramme (or gramme mass).

Action and Reaction. In Expt. II. (p. 119) the ring was found to be in equilibrium under the opposite pulls of the threads *BA* and *CD*. Consider the bit of thread *AB*, it is in equilibrium under the pull ($=W_1$) from *A* upwards and the pull ($=W_1$) from *B* downwards. At *B* the action of the thread on *BC* is equal and opposite to that on *BA*. At every point of the thread a similar argument holds, *i.e.* there are two equal and opposite forces pulling away from each other. This double set of forces is called a **stress**, tensile stress in this particular case.

If a column (Fig. 132) supports a load *W*, then the action of the upper portion on *AP* is (neglecting the weight of the column itself) a downward push $=W$, and the upper part is in equilibrium under the load *W* and the reaction of *AP*. The action at *P*, therefore, on the upper part must consist of an upward push $=W$. Whatever part of the column be considered, the result is the same, at every point there are two equal and opposite pushing forces. This double set of forces is called a **compressive stress.**

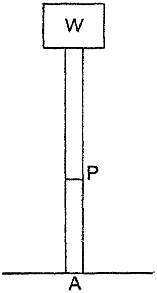

FIG. 132.

No force can be exerted without the presence of an equal and opposite one. If a body be pushed, the body will push back with a force (called the **resistance**) equal in magnitude and opposite to it in sense.

If a spiral spring be pulled out beyond its natural length it tends to shorten and pulls back with a force of equal magnitude. Again, the wind only exerts force in so far as its motion is resisted, and the resisting obstacle reacts on the moving air with a force of equal magnitude.

Put shortly as in **Newton's Third Law of Motion**: the action of one body on another (or of one part of a body on another part) is equal in magnitude and opposite in sense to that of the second body on the first, or still more shortly: **action and reaction are equal in magnitude, have the same axis, but are of opposite sense.**

EXAMPLE. *A book is in equilibrium on a horizontal table, not because there is no force acting on it, but because the pressure of the book on the table, due to its weight, is exactly equal in magnitude and opposite in sense to the reaction of the table on the book.*

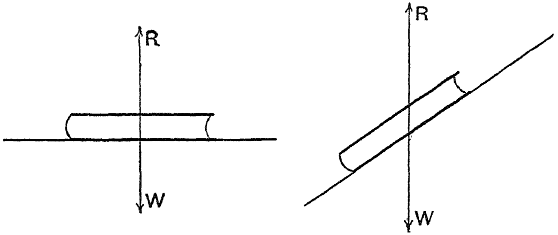

FIG. 133.

Suppose the table to be tilted; then if the book still remains in equilibrium it must be because the reaction of the table is still vertical, and of the same magnitude as before.* The reaction is therefore no longer normal to the table, and hence there must be some force along the common surface of table and book; in fact, there is friction.

Ideal surfaces between which normal action alone is possible are called frictionless or smooth. Smooth as applied to one body only is, strictly speaking, meaningless; it is a term relating to the action and reaction of two bodies. If in any problem one surface is spoken of as being smooth. it is meant that the action between that surface and any other body considered in the problem is wholly normal.

Since the action between any two bodies is never wholly normal, problems involving the supposition that certain surfaces are smooth are to a great extent academic, and the results obtained must be regarded as only first approximations to the real state of things.

Note. In this chapter the weight of a body will be supposed to act through its mass-centre.

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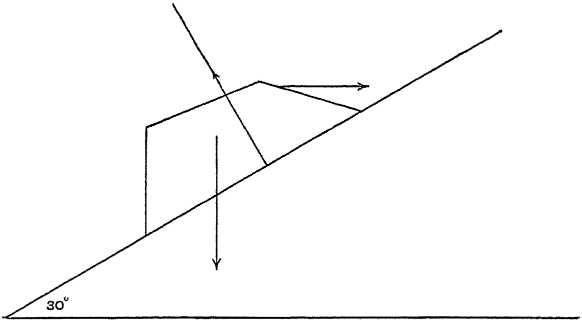
*If a body is in equilibrium under two forces—the weight and the table reaction—these forces can only differ in sense.

EXAMPLE. *A body of weight* W (2·5 *kilogrms.*) *is kept in position on a smooth plane of inclination* 30° *by a horizontal force* α. *What must be the magnitude and sense of* α, *and what is the reaction of the plane?*

The body is in equilibrium under the action of three forces,* viz. the weight, the force α, and the reaction γ of the plane. The latter is perpendicular to the plane, since the plane is smooth.



Set off $AB = 2{\cdot}5''$ (Fig. 134) vertically downwards, draw through A, AC making 30° with the vertical, and through B, BC horizontal.

Then

BC measures the pull
= 1·44 kilogrms.,

and

CA measures the reaction
= 2·89 kilogrms.

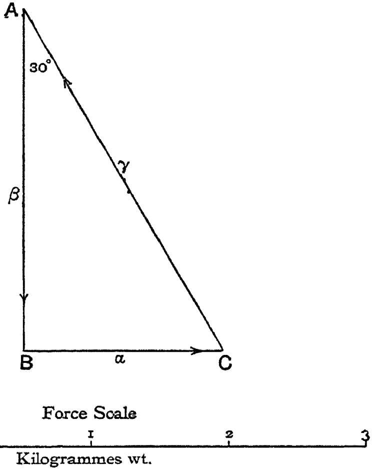


FIG. 134.

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* The axes must be concurrent. p. 132.

(34) If the plane is inclined at 75°, find α and γ.

(35) If the plane is inclined at 30° and the direction of α makes 15° above the horizontal, find α and γ.

(36) If the plane is inclined at 30°, and α's direction is 15° below the horizontal, find α and γ.

(37) Shew from the vector polygon that, whatever the inclination of the plane, the pull will be a minimum if it be applied parallel to the plane.

(38) A garden roller of weight 2 cwts. is hauled up a slope inclined 1 in 5 (1 vertically to 5 horizontally) and held with the handle horizontal. What is the horizontal pull on the handle?

(39) A body weighing 7 cwts. is kept in position on a smooth inclined plane by a force of 2 cwts. parallel to and up the plane and another force inclined at 15° below the horizontal. The ratio of the height and the base of the plane being 0·7, find the force inclined at 30° and the reaction of the plane.

EXAMPLE. *A mass of* 5 *lbs. weight is attached to a string of length* 1 *ft. The string is fastened to a point on the circumference of a smooth, fixed horizontal cylinder of radius* 2 *ft. The point of attachment being* 1·2 *ft. from the top of the cylinder; find the tension in the string and the reaction of the cylinder.*

The direction of the string at C is along the tangent to the circle, the string being supposed quite flexible.

The tension in the string being the same at all points of BC (see formal proof on p. 162) it is immaterial at what point we suppose it fastened to the cylinder; in fact the length of the string may be anything, provided one end is at C and it is wound on the cylinder from the fastened end towards C in a clockwise sense.

Draw a circle of radius 2″ to represent the vertical section of the cylinder containing the weight and string. Step off, from the highest point A (Fig. 135), the arc $AC = 2{\cdot}2''$. Join C to the centre O of the circle. Then draw the vector polygon of the forces; α vertically downwards of length 5 cms., γ parallel and β perpendicular to OC. Then measure γ and β in cms to obtain the reaction of the cylinder and the tension of the string.

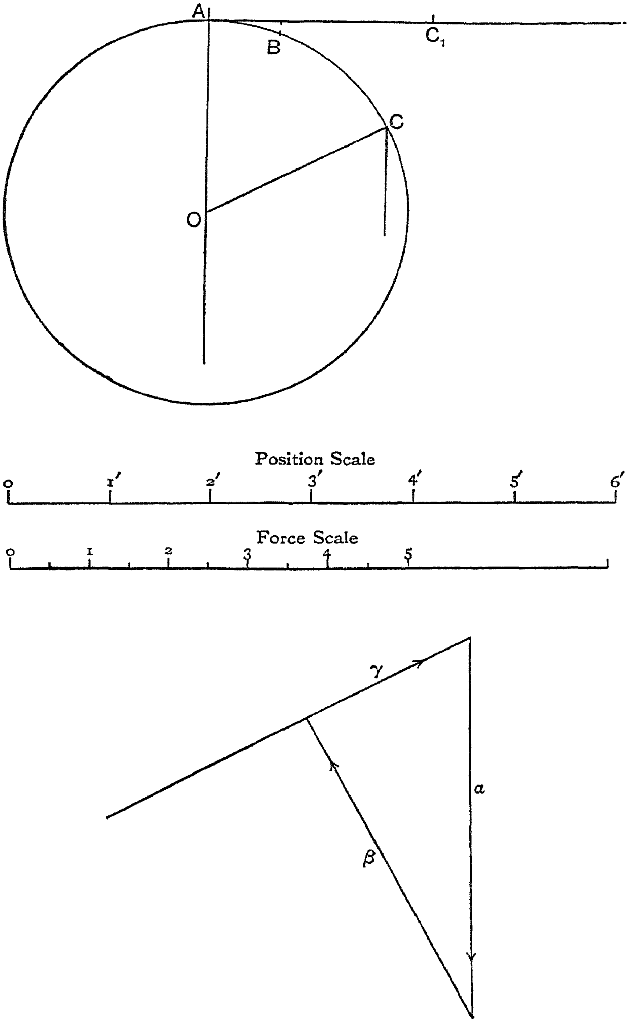
A
B
C₁
C
O
Position Scale
0
1'
2'
3'
4'
5'
6'
Force Scale
0
1
2
3
4
5
γ
a
β

FIG. 135.

EXAMPLE. *A body of* 15 *lbs. weight is sustained on a smooth inclined plane by a horizontal force of* 7·2 *lbs. weight and a force parallel to the plane of* 3·7 *lbs. weight. What is the inclination of the plane and its reaction?*

Draw *OA* (Fig. 136) downwards of length 15 cms., then *AB* horizontally of length 7·2 cms. With *B* as centre, describe a circle of radius 3·7 cms., and, by the aid of set squares, draw a tangent to it from *O*; let *BD* and *OD* be the radius and tangent.

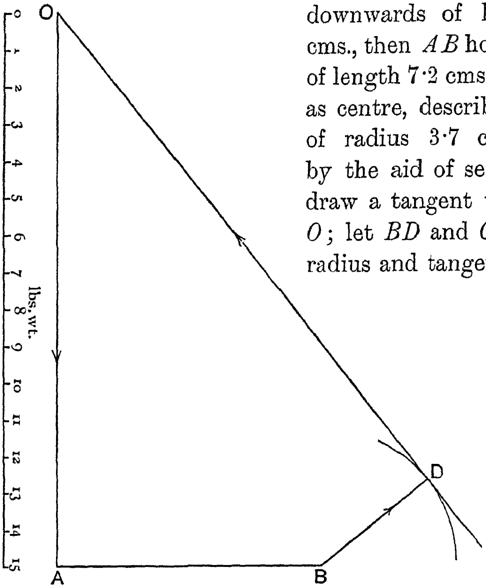

FIG. 136.

Measure *OD* on the cm. scale, this gives the reaction; measure the slope of *DB* by finding how many inches it rises for 1″ horizontally, or use a protractor and obtain the angle *DBA*. (Reaction = 16·25 lbs. and angle of plane 38·2° approximately.)

(40) A mass of 7 lbs. weight is to be attached to the highest point of a smooth horizontal cylinder (radius 2′) by a string which can only bear a tension of 4 lbs.; what is the greatest length of string that may be used? (The reaction being perpendicular to the tension, the vector triangle is right-angled, and since α is known and the magnitude of β, γ is determined.)

(41) Find the inclination of a smooth plane so that a body of 5 kilogrms. weight may be supported on it by a horizontal push of 2 kilogrms. weight.

(42) Find the inclination of a smooth plane so that a body of 17 lbs. weight may be supported on it by a force of 7 lbs. weight applied parallel to the plane. (In this case we know the reaction of the plane is perpendicular to the applied force of 7 lbs., so set off 17 cms. vertically downwards for the weight. From the lower end of this line describe an arc of radius 7 cms. as in Fig. 137, and draw a tangent to it from the upper end. The length of the tangent gives the reaction *R*. The reaction, and therefore the normal to the plane, is now known.)

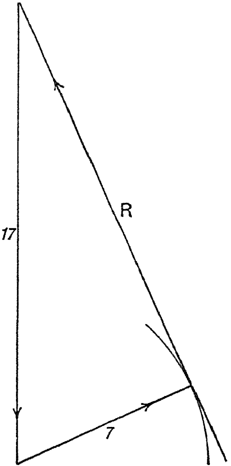

FIG. 137.

(43) A body of weight 15 lbs. is supported on a smooth inclined plane by a horizontal force of 7 lbs. weight together with a force of 4 lbs. weight acting parallel to and up the plane; find the inclination of the plane and the reaction.

(44) A truck weighing 15 cwts. is kept at rest on an incline of 1 in 5 (one vertical to five horizontal) by a rope 6 ft. long attached to the truck 3 ft. above the level of the rails and fastened to a hook midway between them. Find the pull on the rope.

(45) A smooth ring weighing 3 kilogrms. can slide on a vertical circular hoop of radius 2 ft. It is attached to the highest point of the hoop by a string 3 ft. long. Find the tension in the string and the reaction of the hoop on the ring. (The reaction is along a radius of the circle since the ring is smooth.)

(46) A string with equal weights of 11 lbs. attached to its ends is hung over two parallel smooth pegs *A* and *B* in the same horizontal line; find the pressures on the pegs. (The tension of the string is the same throughout; the concurrent forces at each peg which are in equilibrium are the reaction of the peg and the two pulls of the string, one on each side of the peg.)

(47) If in Ex. 46 the line *AB* makes an angle of 40° with the horizontal; find the pressures on the pegs.

(48) A string with equal weights of 750 grms. attached to its ends passes round three pegs in a vertical plane at the vertices of an equilateral triangle. Find the pressures on the pegs when one side of the triangle is horizontal and (i) the third vertex above, (ii) the third vertex below the horizontal side, the string passing under this vertex and over the other two.

Simple Bar Frameworks. In problems on the equilibrium of very simple frameworks of rods we suppose at first that

(i) the weight of the rods may be neglected;

(ii) the joint connecting two rods is made by a perfectly smooth circular pin;

(iii) the loads are applied only at the joints.

The action of the pin on the rod must then pass through the centre of the pin (why?); hence, any rod is under the action of two forces passing through the centres of the end pins, and for equilibrium these forces must be equal and opposite, *i.e.* in the line joining the centres of the pins. The bars may therefore be represented by the lines joining the pin centres.

EXAMPLE. *A wall crane consists of two bars* AC *and* BC *pin-jointed together at* C *and to the wall at* A *and* B. (BC *is called the beam*, AC *the tie rod.*) *A load of* 4·02 *tons is suspended from* C. *Find the stresses in* BC *and* AC *and whether they are tensile or compressive, given that* BC = 10·1 ft., AC = 15 *ft.*

Since the forces on the pin at C are 4·02 tons downwards and pulls or pushes along BC and CA, we have simply to find the forces in the directions CA and CB which will be in equilibrium with 4·02 tons downwards.

Draw first the crane to scale and then set off $OP = 4{\cdot}02$ cms. (Fig. 138) vertically downwards and draw PQ horizontally and QO parallel to AC. The forces at C are given by **OP**, **PQ** and **QO** in magnitude, direction and sense. Scale these vectors; PQ gives 3·64 tons. Notice that OPQ is similar to ABC, hence, if ABC be supposed the vector polygon for the forces at C, then AB represents 4·02 tons. Measure the length of AB, and from this determine the forces represented by BC and CA.

At C the beam BC pushes from left to right, and, therefore, since C is in equilibrium, the pin must push the beam C from right to left and exerts a compressive force on it. Again, the beam is in equilibrium and hence the pin at B must also exert a force on the beam from left to right. Hence, BC is in a state of

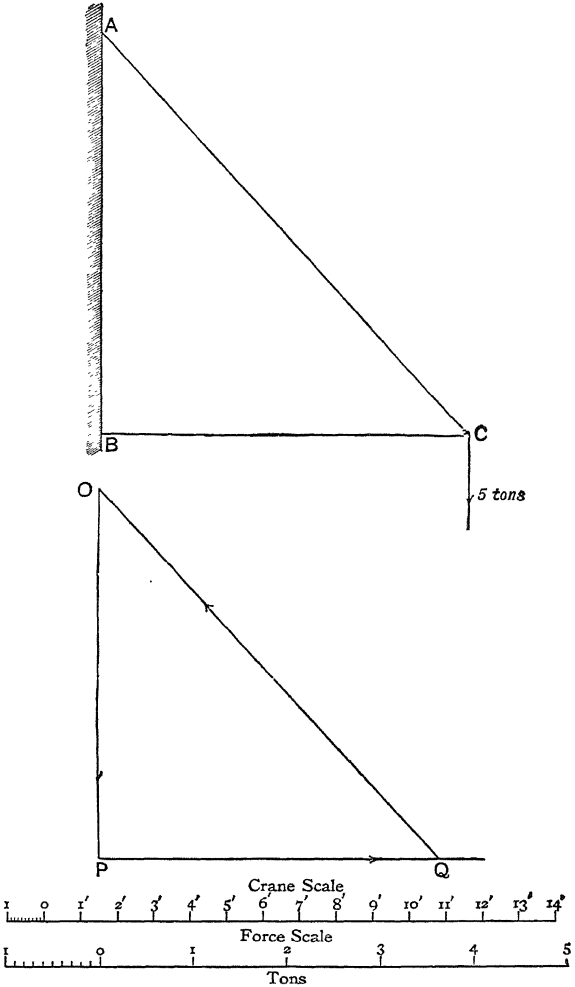
A
B
C
5 tons
O
P
Q
Crane Scale
1 0 1′ 2′ 3′ 4′ 5′ 6′ 7′ 8′ 9′ 10′ 11′ 12′ 13′ 14′
Force Scale
1 0 1 2 3 4 5
Tons

FIG. 138.

compression and the compressive stress is measured simply by the force at either end.

Again, *QO* measures the action of the bar *AC* on *C*, and since it is upwards, the bar evidently pulls at *C*; and further, since the bar is in equilibrium it must also pull the pin at *A*, and hence the bar *AC* is pulled at *C* and *A* with forces tending to lengthen it and must therefore be in a state of tensile stress or (shortly) in tension.

(49) If $BC=9$ ft. and $AC=12$ ft., and the load suspended from *C* is 3·78 tons, find the stresses in *AC*, and *BC*.

(50) If $A\hat{B}C=60°$, $B\hat{C}A=45°$, and the load is 6·3 tons, find the stresses.

(51) If $BC=10$ ft. and $AB=12$ ft., and *BC* slopes downwards at an angle of 30°, find the stresses in *AB* and *BC* due to a load of 2·8 tons wt.

(52) If $BC=10$ ft. and is horizontal, find the stresses in *BC* and *AC* when *AB* has the following lengths 10, 8, 6, 4 and 3 ft. ; the load is 1·7 tons wt. Draw a graph shewing the relation between the length of *AB* and the stress in *BC*.

EXAMPLE. *In a wall crane* ACB (Fig. 139) *the chain bearing the load* W *passes over a smooth pulley at* C *and is fixed to the wall at* E; *find the stresses in* AC *and* CB *given that* AC *is horizontal and of length* 9 *ft.*, BC = 12 *ft.*, AE = 4·4 *ft.*, *and* W = 3·7 *tons.*

Draw the frame to scale, say 1 cm. to a foot. Since the pulley is smooth, the pull on *E*, and therefore the tension in *CE*, is measured by 3·7 tons weight.

Hence, set off $OP=3{\cdot}7''$ vertically downwards to represent the load; then $PQ=3{\cdot}7''$ parallel to *CE*. Through *Q* draw *QR* parallel to *AC*, and through *O* draw *OR* parallel to *BC*; then *OPQR* is the vector polygon of the forces keeping the pin in equilibrium at *C*.

Measure the lines to scale. The senses in which the vectors must be taken at *C* are decided by **OP** and **PQ**. **QR** acts from *A* to *C*, and the bar pushes at *C* and is therefore in compression. **RO** acts from *C* to *B*, and the bar pulls at *C* and is therefore in tension.

(53) Find the stresses when $AE=3$ ft.

(54) Find the stresses if $AC=AB=9$ ft., $AE=4{\cdot}5$ ft., and *AC* slopes upwards at an angle of 20° with the horizontal.

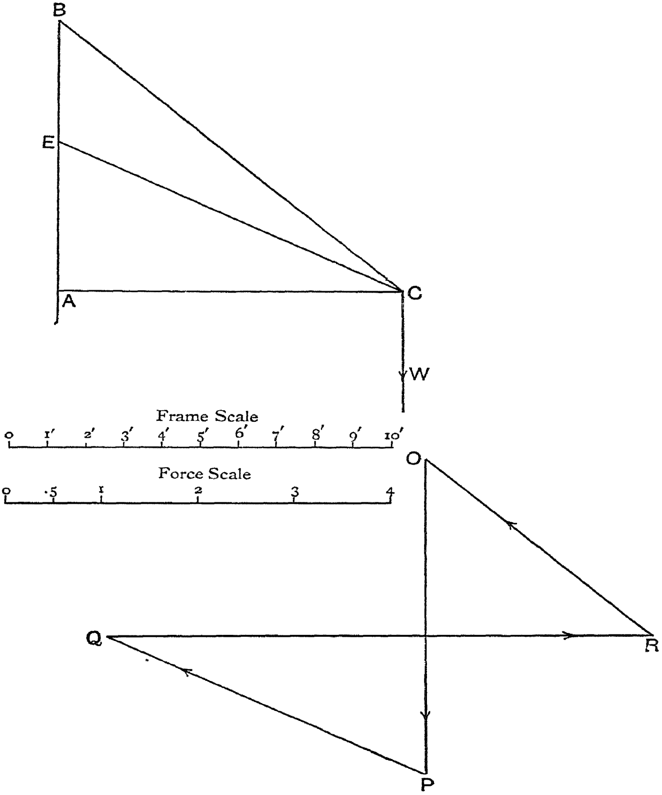
B
E
A
C
W
Frame Scale
0
1′
2′
3′
4′
5′
6′
7′
8′
9′
10′
Force Scale
0
·5
1
2
3
4
O
Q
R
P

Fig. 139.

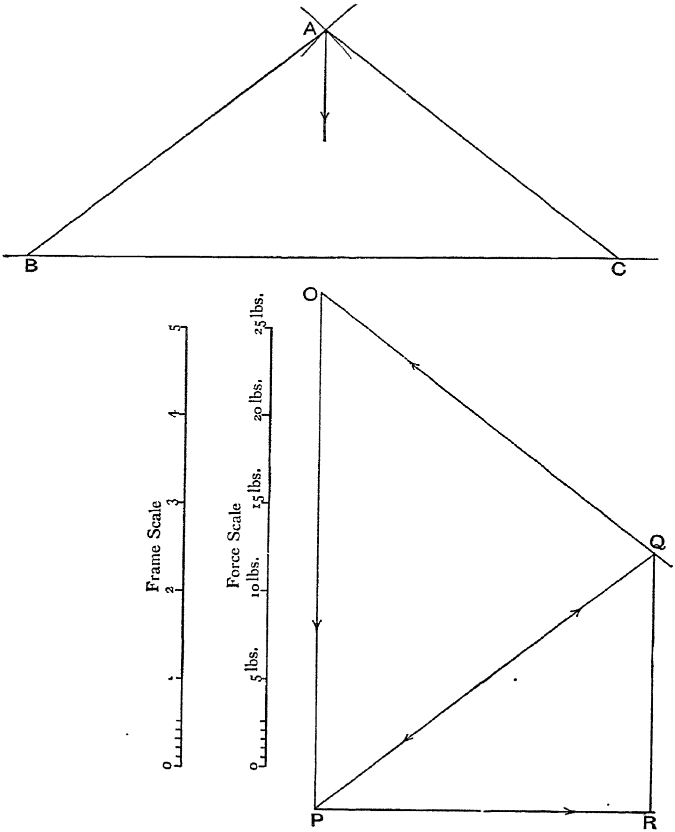
A
B
C
O
Q
P
R
Frame Scale
0
1
2
3
4
5
Force Scale
0
5 lbs.
10 lbs.
15 lbs.
20 lbs.
25 lbs.

FIG. 140.

EXAMPLE. *Two equal rods* AB *and* AC *are pin-jointed together at* A *and their other ends connected by a cord* BC; *the whole rests on a smooth table in a vertical plane with a weight* W = 29·4 *lbs. suspended from* A. *Given* AB = AC = 3·9 *ft. and* BC = 5·9 *ft., find the stresses in* AB, AC *and* BC.

First draw the frame ABC (Fig. 140) to scale (1″ to 1′), and then the vector polygon: $OP = 5{\cdot}88$ inches, PQ parallel to AB, QO parallel to AC. Then these vectors measured on $\frac{1}{5}$″ scale give the stresses in the bars in lbs. weight. Are the rods AB and AC in compression or tension?

For the joint B the force QP pushes. Draw PR parallel to BC and QR vertical, then the sense of the vector triangle for B is QPR, and PR measures the tensile (why tensile?) stress in BC. Why was RQ drawn vertically upwards, and what does it measure?

(55) If $AB = 3$ ft., $AC = 2$ ft., $BC = 3{\cdot}5$ ft. and $W = 2{\cdot}3$ kilogrms., find all the stresses and the reactions of the table at B and C.

(56) The Derrick Crane. BC (Fig. 141) is the post (kept vertical by some means not shewn), AC the jib, AB the tie rod. Given $AB = 6$ ft., $AC = 13$ ft. and $BC = 10$ ft. A load $W = 7{\cdot}4$ tons is suspended from A, find the stresses in AB and AC.

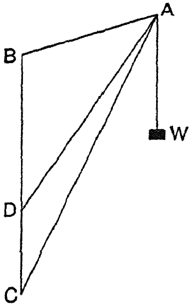

FIG. 141.

(57) If the supporting chain passes over a smooth pulley at A and is fixed at D, where $CD = 3{\cdot}5$ ft., find the stresses in AB and AC.

(58) Given $AB = 17$ ft., $AC = 25$ ft., $BC = 16$ ft., $CD = 5$ ft. and $W = 14{\cdot}5$ tons, find the stresses in AB and AC.

(59) A picture weighing 6·5 lbs. is hung by a wire over a smooth nail. If the distance apart of the points AB at which the wire is fastened be 1 ft. 7 in. and the length of the string 2 ft. 3 in., find the pressure on the nail and the tension in the string.

(60) If the length of the string in Ex. 59 vary, draw a graph shewing the relation between the tension of the string and its length.

(61) Light rods AC, CB, of lengths 7·2 ft. and 5·7 ft. are pin-jointed together and to two fixed points A and B distant 6·3 ft. apart. AB is inclined to the horizontal at an angle of 25° (A being the higher) a load of 4·7 cwts. is suspended from the pin joining the two rods; find the stresses in the rods.

(62) *ABC* (Fig. 142) is a wall crane pin-jointed at *A*, *B* and *C*; a load *W* of 5 tons is suspended from a pulley *D*, which is attached to the crane at *B* and *C* by a chain *BDC*. $ABD = 72°$, $AC = 7{\cdot}5$ ft., $AB = 12{\cdot}9$ ft.; find the stresses in *AB* and *BC*. (Notice that *BD* and *CD* must be equally inclined to the vertical, since the tension throughout the chain is constant. Hence first draw the vector triangle for the forces at *D*, and determine this tension. Knowing the pull of the chain at *B*, the stresses in *AB* and *BC* can be found.)

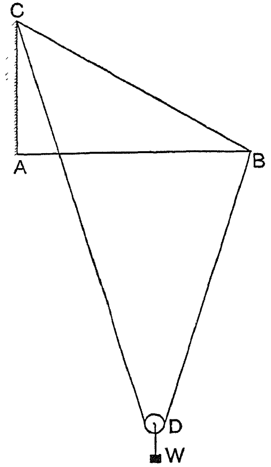

FIG. 142.

(63) A picture, of weight 11 lbs., is suspended from a smooth nail by a continuous string passing through two smooth rings on the picture frame distant apart 1 ft. 7 in. If the height of the nail above the two rings be 3 ft., find the tension in the string and the pressures on the nail and rings.

(64) *AB* and *BC* (Fig. 143) are light rods pin-jointed together at *B*, and to fixed points at *A* and *C*. A load *W* (7 cwts.) is suspended by a chain which passes over a smooth pulley at *B* and is attached to *M*, the mid-point of *AC*. The load is pulled by a horizontal rope until the chain makes an angle of 30° with the vertical. Find the stresses in the rods and chain, given that $AB = 10$ ft., $BC = 5{\cdot}4$ ft. and $AC = 6{\cdot}22$ ft.

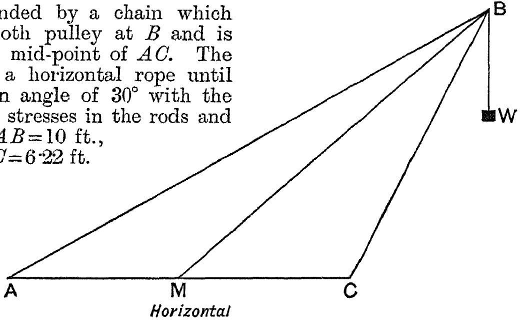

FIG. 143.

Components of a Force. In relation to their resultant the forces of a given system are called the **components.** Finding the resultant of a set of concurrent forces is a unique process; the converse problem of finding the components when the resultant is known is not in general unique.

A force may be decomposed into two components having given directions and passing through any point on the axis, in one and only way only.

The proof is exactly the same as that for the decomposition of vectors, on p. 84.

When **the component of a force in a given direction** is spoken of without reference to the other component, it is always implied that **the two components are perpendicular.**

Scalar Conditions of Equilibrium for Concurrent Forces. For equilibrium under concurrent forces, it is a sufficient and necessary condition that the vector polygon of the forces should be a closed figure.

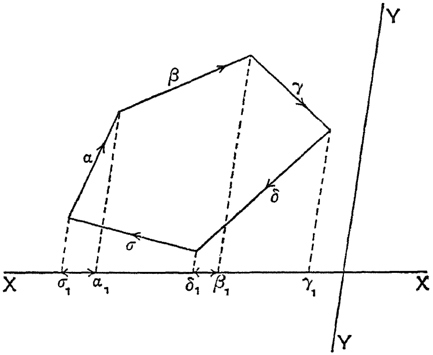

FIG. 144.

Let the axes of the forces be supposed concurrent, and let α, β, ... σ be their vectors whose sum is zero (*i.e.* the vector polygon is closed). Draw any line XX; project the vectors on to this line by drawing parallels through the end points of the vectors. If α_1, β_1, ... be the projections, Fig. 144 shews that

$$\alpha_1 + \beta_1 + \gamma_1 + \delta_1 + \sigma_1 = O.$$

Similarly project on YY a line parallel to the former direction of projection, and establish a similar theorem for the projection on it, viz.

$$\alpha_2 + \beta_2 + \gamma_2 + \delta_2 + \sigma_2 = 0.$$

Then α_1 and α_2 are the components of α in the directions XX and YY, and so for the other components, and the sum of the components in any two directions is zero.

Conversely, if the sum is zero in any two directions the vector polygon is closed, and the forces (if concurrent) are in equilibrium. One direction is not sufficient, for it might happen, as in Fig. 145, that though the polygon is not closed, the first and last points of the projections are coincident

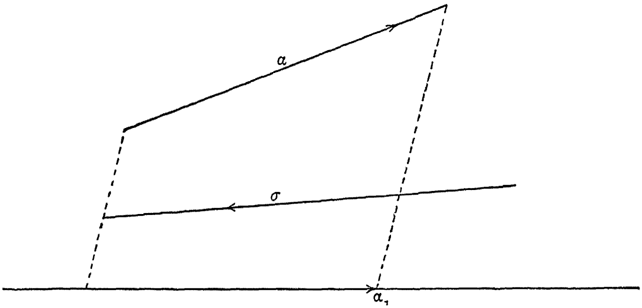

FIG. 145.

The two directions being at right angles we have the theorem: **The sum of the components in any direction of all the forces acting on a body in equilibrium is zero.**

Again, $-\sigma$ is the resultant of $\alpha + \beta + \gamma + \delta$, and hence we get the theorem: **The sum of the components in any direction of any number of concurrent forces is equal to the component of the resultant in that direction.**

(65) Five concurrent forces in a horizontal plane have components 3·7, 2·1, 1·8, 1·7 and 2·9 towards the N., and components 1·2, 3·7, 2·4, 3 and 3·2 towards the E. Find the resultant in magnitude, direction, sense and position.

(66) Mark on squared paper the four points whose coordinates are (1, 0), (1·7, 2), (2·3, 1), (3·2, 4) inches, and let the lines joining the origin to these points represent concurrent forces to the scale of 1 cm. to a kilogrm. Find the resultant by (i) the vector polygon method, (ii) the component method.

(67) Forces of magnitude 5, 2·8, 3·1, 4·7 are concurrent and make angles of 15°, 30°, 60° and 75° with a line through the point of concurrence. Find the forces in this line and a line perpendicular to it which would be in equilibrium with the given forces.

EXAMPLE 1. *A horse begins to pull a small tramcar with a force* P = 500 *lbs. weight. The traces make an angle of* 25° *with the horizontal, find the component of* P *in the direction of motion. If the weight of the car be* $\frac{1}{2}$ *a ton, what is the reaction of the ground? (Suppose no friction.)*

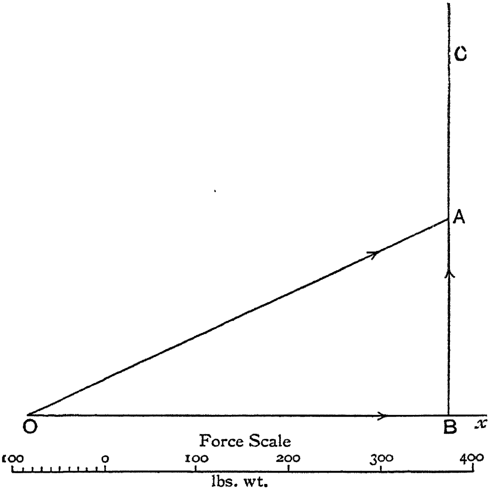

FIG. 146.

Draw $OA = 5''$ (Fig. 146) making 25° with Ox, and draw AB perpendicular to Ox; then, since as vectors $\mathbf{OA} = \mathbf{OB} + \mathbf{BA}$, $\mathbf{OB}$ represents the forward pull on the car.

From B along BA set up $BC = 11{\cdot}2''$, then $\mathbf{AC}$ gives the reaction of the ground, for it represents the weight of the car, less the vertically upwards component of the pull of the traces.

EXAMPLE 2. *The points* A *and* B *are* 5″ *apart and distant* 1″ *and* 3·7″ *respectively from the line* CD, *and both on the same side of it. Find the components through* A *and* B *of a force of* 8 *lbs. in* CD, *when* (i) *the component through* A *is perpendicular to* CD; (ii) *the components through* A *and* B *are mutually perpendicular*; (iii) *the components are equal in magnitude.*

(i) Draw, through A, AO perpendicular to CD. Join BO, and find the components of the 8 lbs. weight along OA and OB.

(ii) Draw a semicircle on AB, cutting CD in O and O_1; then OA, OB, O_1A and O_1B are possible directions for the components. There are thus in this particular case two sets of components which will satisfy the conditions of the problem. Find these components.

(iii) Draw AM perpendicular to CD and produce it to A_1, where $AM=MA_1$. Join A_1B, cutting CD in N; then AN and NB are the required directions. Find the components in these directions.

(68) In the above example, if B is on the opposite side of CD to A, determine the components in the three cases (i), (ii) and (iii).

(69) The pressure of wind on a sail when the sail is perpendicular to the wind is 500 lbs. weight; find the normal pressure on the sail when the wind makes angles of 15°, 40°, 65° and 75° respectively with the sail.

(70) Find the components of a force of 11 lbs. weight making angles of 30° and 75° with it.

(71) A force of 17 lbs. weight is directed due N.; find the components in the directions (i) N.E. and N.W., (ii) E. and N.W., (iii) S.E. and 30° W. of N.

(72) Two ropes are attached to the coupling of a railway van and are pulled horizontally with forces of 200 lbs. and 270 lbs. weight. The lengths of the taut ropes are 18 ft. and 21 ft. and their ends remote from the truck are at distances of 10 ft. and 7 ft. respectively from the centre line of the rails. Find the forward pull of the van and the side thrusts on the rails when (i) both ropes are on the same side, (ii) on opposite sides of the rails.

(73) On squared paper mark the positions of two points whose coordinates are (1, 2) and (2·4, 1·2) inches; find the components through the origin and these points of a force of 7 lbs. weight acting (i) along the axis of x, (ii) along the axis of y, (iii) along the line bisecting the angle xOy.

(74) On squared paper mark the point whose coordinates are (2·7, 1·1), and draw a line parallel to Oy and distant 1″ from it on the negative side. Find the components of a force of 5 lbs. weight, one of which is along this parallel and the other passes through the given point when the axis of the

force is (i) along Ox, (ii) along Oy, (iii) a line making 30° with Ox and cuts Ox at 1·5″ from the origin on the positive side.

(75) A smooth inclined plane (Fig. 147) rising 1 in 3 has a smoothly running pulley at the top. A body of weight W (11·6 lbs.) is kept in equilibrium by the pull of a string parallel to the plane. The pull being produced by a freely hanging weight P, find P and the reaction of the plane.

What is the component of the weight parallel to the plane? What is the vertical component of the reaction of the plane?

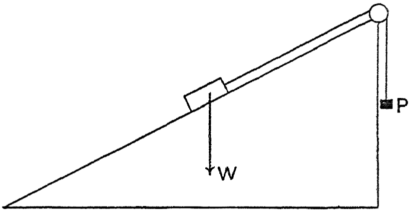

Fig. 147.

(76) A man distant 13·5 ft. from a tree pulls at the upper part of the trunk by a rope of length 40 ft. His pull is equal to a weight of 80 lbs. What is the horizontal pull on the tree, and what is the force producing compressive stress in the trunk?

(77) A barge is towed by a horse with a pull P of 152 lbs. weight making an angle of 20° with the direction of the bank. What is the force producing forward motion, and what would be the side thrust of the water on the barge if there were no side motion?

(78) A block is partly supported by a smooth right-angled wedge of weight 18 lbs. (as in Fig. 148) the height and base of the wedge being 3·2 ft. and 6 ft. respectively. If, to maintain equilibrium, the wedge has to be pushed with a horizontal force of 28 lbs. weight, what are the reactions of the wedge on the body and on the horizontal table?

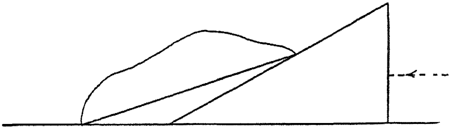

Fig. 148.

(79) A uniform cylinder of weight 57 lbs. rests on two inclined planes as indicated in Fig. 149. The planes are hinged together at A; what is the tension at A, and what is the pressure of each plane on the ground, given that the wedges are equal in all respects, each weighing 15 lbs., and that

$$\frac{BC}{AB}=\frac{3}{7}?$$

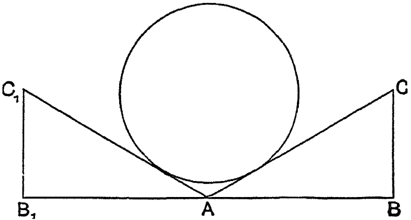

Fig. 149.

EXAMPLE. OF (Fig. 150) *represents the crank of an engine*, F *moving in the circle* DFE *and* O *being fixed.* CF *is the connecting rod*, C *being the cross head of the piston rod which moves to and fro along* AB (AB = DE). C *is kept in the line* AB *by guides. The forward thrust on* C *being* 5000 *lbs. weight, find the force transmitted along the connecting rod* CF *and the side pressure on the guides at* C *(assuming no friction) given that* CF = 6·5 *ft.*, DE = 3 *ft. and* AC = 6″ *the direction of motion* F *being as indicated.*

Find also the components of the force transmitted along CF *in the direction of the forward motion of* F *and perpendicular to it, (i.e., along the tangent and radius at* F*).*

First draw the position diagram to scale, say 2 cms. to 1 ft. Next construct the vector diagram $PQ = 5''$ to represent 5000 lbs.; then QR and PR are perpendicular to AB and parallel to CF respectively. **QR** is the thrust on the guides (and **RQ** is the reaction of the guide on C keeping it in the path ACB) and **PR** is the force transmitted along the connecting rod.

Draw PS perpendicular and RS parallel to OF, then **PR** acting along CF is equivalent to **PS** acting along the tangent at F and **SR** acting from F to O. **PS** then gives the forward thrust of F.

(80) Find the force on F urging it round the circle when $AC = 0{\cdot}2$, $0{\cdot}4$, $0{\cdot}6$ and $0{\cdot}8$ times AB.

EXAMPLE. *Draw a graph shewing the connection between the position of* C (*Fig.* 150) *and the thrust on* F *urging it round the circle.*

Divide AB into ten equal parts and draw ordinates at A and B and the points of division. Produce CF to cut the ordinate at O in G. Project G horizontally on to the ordinate at C. Do this for the eleven marked positions of C and join the points so determined by a smooth curve. The force scale for this representation is OF to 5000 lbs.

Compare the results with those obtained by the vector polygon.

* **Proof.**

From the construction of Fig. 150 we have the following relations between the angles :

$$RPQ = GCO = 90^\circ - OGF, \qquad PRS = GFO,$$

and hence $$\frac{PQ}{PR} = \cos RPQ = \cos (90^\circ - OGF) = \sin OGF,$$

$$\frac{PS}{PR} = \sin PRS = \sin GFO;$$

$$\therefore \frac{PS}{PQ} = \frac{\sin GFO}{\sin OGF} = \frac{OG}{OF}.$$

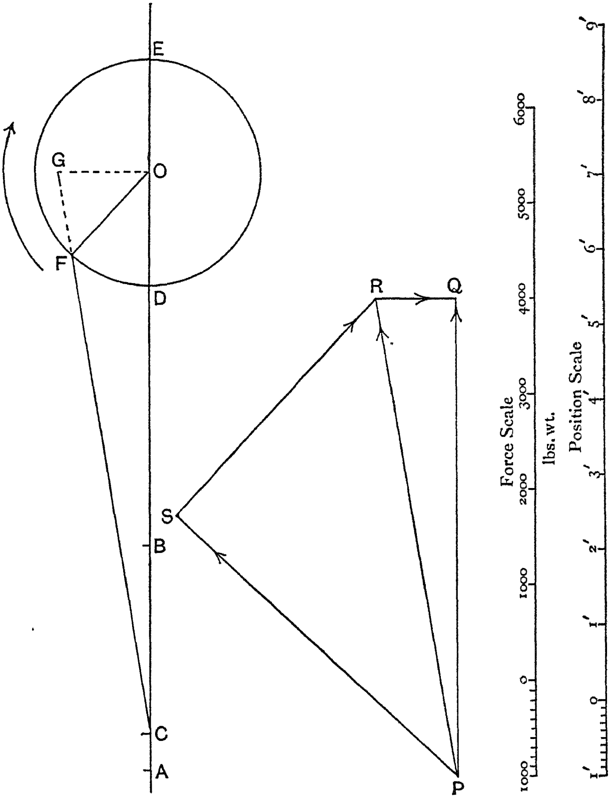
E
G
O
F
D
R
Q
S
B
C
A
P
Force Scale
lbs. wt.
Position Scale
1000
0
1000
2000
3000
4000
5000
6000
1'
0
1'
2'
3'
4'
5'
6'
7'
8'
9'

Fig. 150.

Hence, if OF be taken to represent PQ or 5000 lbs. wt., OG will represent PS or the forward thrust on the piston.

*(81) If the force on the piston decreases uniformly from 5000 lbs. at A to zero at B, find by a graphical construction the forward thrust on F when the cross head is at C and $AC=0{\cdot}4AB$.

*(82) Construct the curve giving the relation between the forward thrust on F and the displacement of C for the variable force given in the last example. Set up, perpendicular to AB, $AA_1=$radius of crank circle and project from G to G_1 on AA_1 and join G_1B; the point of intersection of G_1B and CC_1 gives the force required for displacement AC.

EXAMPLE. AB (Fig. 151) *represents a sail of a ship whose keel line is as shewn. The thrust* α *of the wind on* AB *if perpendicular to the wind would be* 500 *lbs. weight. If* AB *makes an angle of* 30° *with the keel line and the relative velocity of the wind to the ship be in direction* CM, *making* 45° *with keel line, find the thrust urging the ship forward.*

Resolve α into β and γ perpendicular and parallel to the sail AB. γ has no effect on the ship's motion. Find the components of β along and perpendicular to the keel line; the former, δ (approximately 64·7 lbs. weight) is the thrust urging the ship forward, the latter, ϵ, tends to produce lee-way and in good sailers is nearly balanced by the resistance of the water to side motion and the force of the current on the rudder.

In the vector diagram, since RPQ is a right angle, a circle described on RQ as diameter will pass through P. Draw this circle. As the direction of the sail line AB is changed the point P will move on this circle. Evidently as P changes, the length QT will alter and it will be greatest when PT is a tangent to the circle.

Now, the radius being perpendicular to the tangent at any point, the line joining P (when PT is a tangent) to the mid-point S of RQ must be perpendicular to PT and therefore parallel to the keel line. Hence, to find the best position for the sail, bisect RQ at S and describe a circle of radius SQ; then draw SP parallel to the keel line cutting the circle at P. RP gives the direction in which the sail should be set and the greatest possible forward thrust is given by QT. Since $S\hat{R}P=\frac{1}{2}Q\hat{S}P$ we may give this direction as the one bisecting the angle between the keel line and the direction of the relative wind.

No matter how small the angle between the keel line and the wind direction, there will always be a force urging the vessel on. If there be much lee-way, sailing close to the wind is impossible.

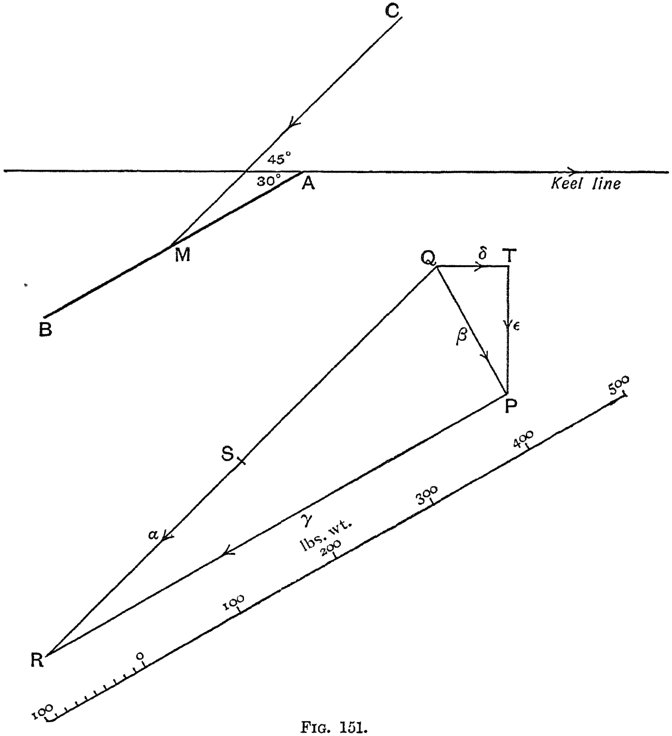

FIG. 151.

(83) Draw a figure for the case when the sail is set on the other side of the keel line, and shew that this case is an impossible one.

(84) The keel line being from W. to E. and the relative wind from the N.W.; find the forward thrust on the ship when the sail is set 25° S. of W.

(85) Shew from the vector diagram for given directions of the keel line and stern wind that the greatest forward thrust would not be obtained by putting the sail as nearly perpendicular as possible to the keel line.

(86) The keel line being from N. to S. and the relative wind from E. to W., find the forward thrust when the sail makes an angle of 20° with the keel line. Find the angle at which it should be set to give maximum forward thrust on the ship.

(87) The force of a current on a rudder when placed perpendicular to the stream is 50 lbs.; find the retarding force on the ship when the rudder makes angles of 20°, 30° and 60° with the keel line. Shew that, in a race, the rudder should be used as little as possible.

(88) Explain how it is that a kite, though fairly heavy, is enabled to rise in the air.

(89) The force of the wind on a kite if placed perpendicular to it would be 5 lbs. When the kite makes an angle of 35° with the horizontal, find the force due to the wind urging it upwards.

(90) In Ex. 89 if the kite be stationary and its weight 10 ozs., what is the pull of the string on the kite in magnitude, direction, and sense.

Body in Equilibrium under Three Non-Parallel Forces. Experiment VII. on p. 122 shewed that when a body is in equilibrium under three non-parallel forces, the axes of the forces are concurrent. The same result follows from the combination of concurrent forces, deduced from Newton's Second Law of Motion, since equilibrium is only possible under three forces when the resultant of any two differs only in sense from the third.

This consideration enables us to draw the axes of those forces in equilibrium when one force is unknown in direction.

EXAMPLE. *A uniform beam rests with one end against a smooth vertical wall and the other on rough ground. Determine the reactions of the ground and wall.*

AB (Fig. 152) is the beam of length 25 ft., $A\hat{B}C = 60°$ and the weight is 0·505 cwt.

Draw the beam in position (scale 1 in. to 5 ft.), then draw a vertical through G the M.C. of beam, and a horizontal through A, intersecting in O. Join OB, then BO is the direction of the ground's reaction.

Construct the vector polygon (scale 10 cms. to $\frac{1}{2}$ cwt.). Draw $PQ = 10$ cms. downwards; then QR horizontal and RP parallel to BO. Scale the lengths, QR and PR giving the reactions ($QR = 0·145$ cwt., $RP = 0·525$ cwt. approximately). Why is AO drawn perpendicular to AC?

(91) Determine the reactions of the wall and ground if AB is inclined at 45° to the horizontal.

(92) Determine the reaction of the wall and ground if AB is inclined at 40° to the horizontal and the mass centre G of the beam is at 9 ft. from the ground, reckoned along the beam.

(93) A uniform beam *AB* hinged at *A*, is supported in an inclined position to a vertical wall *AC* by a string *CB* fixed to the wall at *C*. The weight of the beam is 17 kilogrms., $AB=4$ ft., $AC=2$ ft., $BC=5{\cdot}2$ ft., find the tension in *BC* and the reaction at *A* on the beam.

(94) With dimensions as in Ex. 93, find the stress in *BC* if the mass-centre of the beam be $\frac{1}{3}$ *AB* from *A*.

(95) Draw a graph shewing the connection between the distance of *G* from *B* and the tension of the string *BC*.

(96) In Ex. 94, if $AB=30$ ft. and $ABC=75°$, find the reactions.

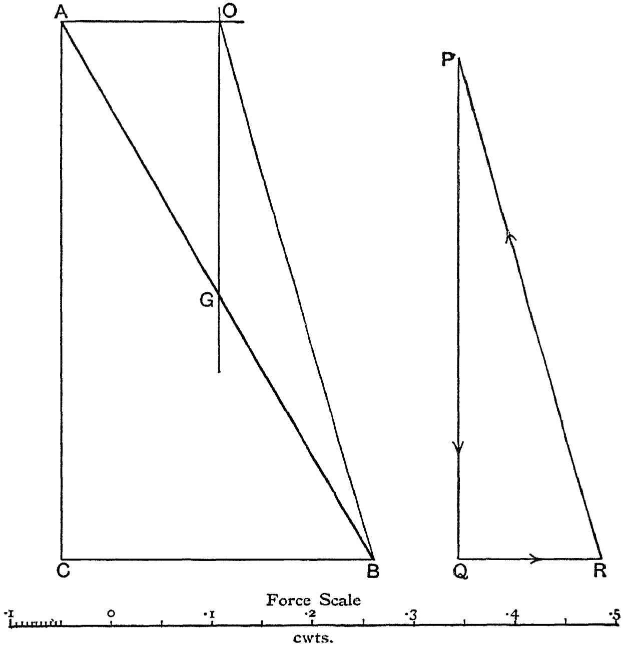

Fig. 152.

(97) A uniform beam *AB* of length 25 ft. and weight 70 lbs. is hinged to a wall at *A* (19 ft. above the ground at *O*), the other end *B* rests on a smooth inclined plane *OB*. Find the reactions at *A* and *B* when the inclination of the plane is 30°, 15° and 60° respectively.

(98) Draw the two sides and base of a rectangle, the sides being 3″ and base 2″; draw a diagonal and produce it 4·5″. Let the two sides of the rectangle represent vertical boards securely fixed in the ground (base), and the diagonal produced a uniform beam. The beam being smooth, find the reactions at its points of contact: weight of beam 2 cwts.

(99) A uniform beam of length 3 ft. is hinged at one end to the lowest point of a horizontal hollow circular cylinder of inner radius 2·5 ft. The other end of the beam rests against the inner smooth surface of the cylinder in a plane perpendicular to the axis of the cylinder. Find the reactions at the two ends of the beam, the weight of the beam being 530 lbs.

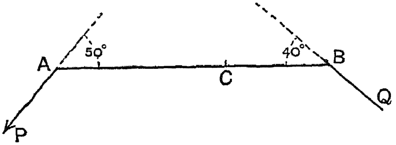

FIG. 153.

(100) AB (Fig. 153) is a weightless rod 5 ft. long, which can turn about C as a fulcrum; $AC=3{\cdot}2$ ft.; it is acted on by two forces P and Q as shewn. $P=100$ lbs.; find Q and the reaction at C.

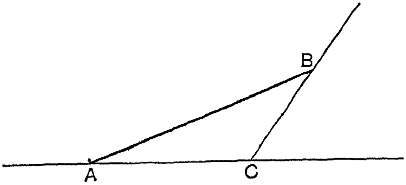

FIG. 154.

(101) A uniform beam AB (Fig. 154), 13 ft. long and of weight 80 lbs., rests against a smooth inclined plane BC (rising 4 ft. vertically to 7 ft. horizontally) and is prevented from sliding by a peg at A, $AC=9$ ft. Find the reaction of the plane and the total reaction at A.

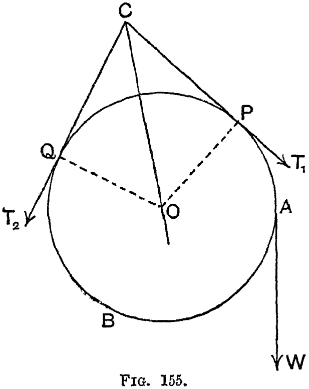

FIG. 155.

The Smooth Pulley. A flexible string of negligible weight is fastened at B to a smooth pulley (Fig. 155) and passing over it bears a load W. Consider the equilibrium of any part QP of the string. The forces acting are the pulls (tensions) at Q and P and the reactions of the surface QP. The former are tangential and their axes intersect at C, the latter are normal and have therefore a resultant passing through *O*. Since three forces in equilibrium must be concurrent, the resultant reaction must pass through *C* as well as *O*. But *CO* bisects the angle *QCP*, and hence, from a trial stress diagram, we see that the tensions at *Q* and *P* must be equal in magnitude.

In some problems the axes of one, two, or more of the forces are unknown in direction, but other geometrical conditions are given which, with the aid of a trial diagram, will enable the solution to be found.

EXAMPLE 1. *A uniform heavy rod, of weight* 9 *lbs. and length* 3 *ft., is suspended from a point by two strings of length* 2·5 *and* 2 *ft. respectively attached to its ends. Find the equilibrium position, and the stresses in the strings.*

If *AB*, *AC*, *BC* (Fig. 156) be the two strings and the rod, then on the system of strings and rod act two forces, the weight of the rod at *M*, its mid-point, and the reaction at *A*. These must be in a line, since there is equilibrium, and hence *AM* must be vertical. Draw the triangle *ABC*, in any position, and its median *AM*. Then, if *AM* be vertical we have the required position, and the vector polygon can be drawn. (The usual convention in books is to represent the vertical in space by a line parallel to the bound edge of the paper; if this convention be adhered to, another triangle $A_1B_1C_1$ must be drawn with *AM* vertical, and this can easily be done by constructing a parallelogram, whose diagonal is vertical and equal to

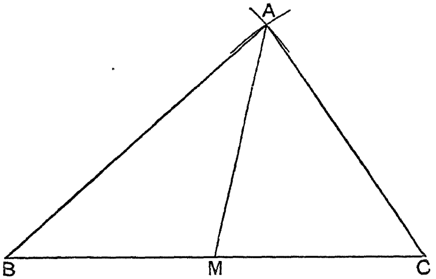

FIG. 156.

$2AM$, and whose adjacent sides are equal to AB and AC.) Complete the solution.

*EXAMPLE 2. *A heavy uniform smooth ring weighing* 17 *lbs. slides on a string of length* 4·92 *ft.; the ends of the string are fastened to two hooks* A *and* B, *whose distance apart is* 3·95 *ft.*, A *being* 0·98 *ft. above* B; *find the position of equilibrium and the tension of the string.*

Since the ring is smooth the tension of the string must be the same on both sides, and hence from a trial stress diagram we see that the two parts of the string must be equally inclined to the vertical. Draw ADB (Fig. 157), where $AD = 0{\cdot}98''$ and is vertical, $AB = 3{\cdot}95''$ and DB is horizontal. With B as centre, describe a circular arc of radius $4{\cdot}92''$ cutting AD produced in E, or set an inch scale so that $BE = 4{\cdot}92''$. Bisect AE at M and draw MN parallel to DB, then ANB is the form assumed by the string.

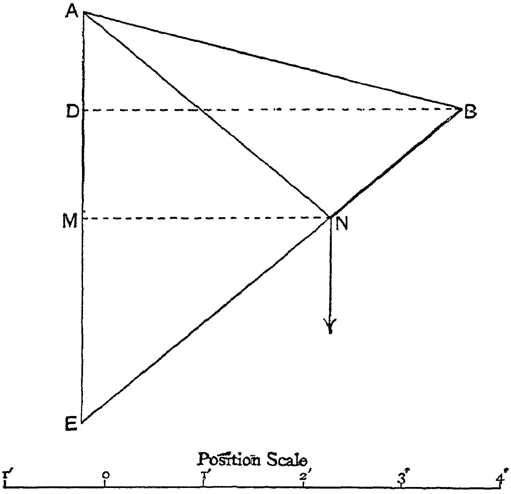

FIG. 157.

For $AN = EN$, and therefore $AN + NB = 4{\cdot}92''$, and AN and NB make equal angles with the horizontal. To find the tension, draw, in the vector diagram, lines parallel to *AN* and *NB* from the extremities of the vector, giving the load of 17 lbs.; complete the solution.

*EXAMPLE 3. *A uniform beam weighing* 105 *lbs. rests with one end* A *in contact with a smooth plane of inclination* 35°, *the other end* B *rests on a smooth plane of inclination* 50°. *Determine the reactions of the supporting planes and the position of the beam.*

Draw first the vector polygon of the forces, *PQ* representing 105 lbs. weight to scale, then draw *QR* and *PR* making angles of 31° and 50° with *PQ*. These lines give the reactions at *A* and *B*. Draw two lines *AC* and *BC* for the planes and at any two points *A* and *B*, their normals intersecting in *O*. Join *O* to the mid-point *M* of *AB*. Complete the parallelogram *OATB* of which *OA* and *OB* are adjacent sides. Then *OTB* and *OAT* should be similar to *DRQ*; hence if *PQ* be bisected at *S*, *CB* should be parallel to *SR*, hence *SR* gives the inclination of the beam.

(102) A rod of length 7″ lies in a smooth hollow horizontal cylinder, perpendicular to its axis, of radius 9″. The mass-centre of the rod is at a point distant 2·5″ from one end; draw the position of the rod in the bowl when in equilibrium, and measure its slope.

NOTE. The M.C. of the rod must be vertically under the axis of the cylinder.

*(103) A uniform rod, of weight 7 kilogrms., can turn freely about one end in a vertical plane; it is pulled by a horizontal force of 4·3 kilogrms. weight at its free end. Draw the rod in its position of equilibrium, and measure its slope.

NOTE. Three forces act on the rod and must pass through a point; knowing the vertical and horizontal forces, the reaction at the hinge can be found. From any point *O* draw two lines: (i) *OB* parallel to the reaction, and (ii) *OA* horizontally. Bisect *OA* at *M*, and draw verticals from *M* and *A*. Where the former cuts *OB* (at *B* say) draw *BT*, cutting the vertical through *A* in *T*, then *OT* is the direction of the beam. Measure *OT* in cms. or inches, and determine the scale to which the figure is drawn.

*(104) Solve the previous exercise if the rod is not uniform and the M.C. is at a distance of $\frac{1}{3}$ of the length from the lower end.

*(105) A rod of length 6 ft. has its M.C. at a distance of 2 ft. from the end which rests on a smooth plane of inclination 30°, the other end rests on another smooth plane whose inclination is 45°. Draw the rod in its position of equilibrium; its weight being 5 cwts., find the reactions of the planes.

*(106) A uniform beam *AB* (Fig. 158), of length 7 ft. and weight 27 kilogrms., can turn freely, in a vertical plane, about *A*; to its upper extremity *B* is fastened a cord which runs over a smooth pulley at *C*, 9 ft. vertically above *A*, and carries a weight of 11 kilogrms. Find the position of the beam and the reaction at the hinge.

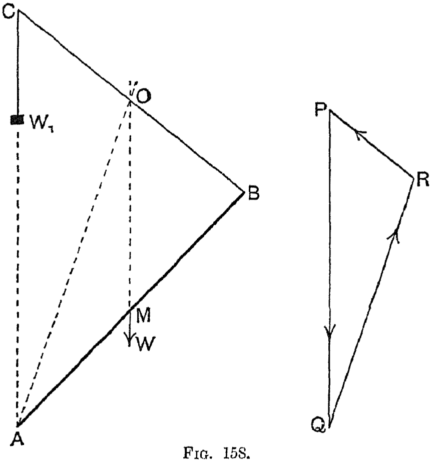

Fig. 158.

Draw a trial figure *ABC* and a stress diagram *PQR*, where *PQ* represents the weight of the beam, *RP* the tension of the cord (=11 kilogrms.) and *QR* the reaction at *A*. Then *PQR* should evidently be similar to *AOC* (*O* being point of concurrence of the axes of the forces). Hence

$$\frac{CO}{CA}=\frac{PR}{PQ}=\frac{11}{27},$$

and hence *CO* and therefore *CB* (=2*CO*) is known, and hence the triangle *ACB* can be constructed to scale. Do this construction and determine *QR*.

MISCELLANEOUS EXAMPLES. IV.

1. With the aid of your instruments find the resultant of the two forces represented in magnitude and direction by the straight lines shewn in the

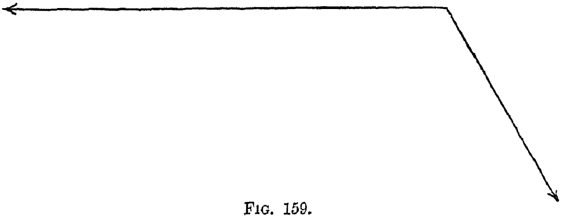

Fig. 159.

diagram (Fig. 159). Assuming that one inch represents 10 lbs. weight, write down the magnitude of the resultant. Also, express in degrees the angle the resultant makes with the greater of the two forces. (Engineer Students, 1903.)

2. Draw diagrams to shew the directions in which each of the following sets of forces must act so as to maintain equilibrium, if they can do so.

Set A. 3, 4, 5. Set B. 1, 1, 3. Set C. 4, 1, 3.

(Naval Cadets, 1904.)

3. Two men, who are lifting by ropes a block of wood, exert pulls of 45 lbs. and 65 lbs. respectively. The ropes are in the same vertical plane; the rope to which the smaller pull is applied makes an angle of 25° with the vertical, and the rope to which the other pull is applied makes an angle of 33° with the vertical on the opposite side. Determine graphically, or in any other way, the actual weight of the block of wood if it is just lifted by these two men. (Naval Cadets, 1904.)

4. One of two forces, which act at a point, is represented numerically by 7; the resultant is 14 and makes an angle of 30° with the force of 7; find graphically the magnitude and line of action of the second force. Also calculate the magnitude to two places of decimals, and measure the angle between the two forces as accurately as you can.

5. Three forces, acting in given directions, are in equilibrium at a point; shew how to find the relative magnitudes of the forces. What additional information suffices for the determination of the absolute magnitudes?

Two small equal brass balls, each weighing $\frac{1}{10}$ oz., are suspended by equal silk threads, 12 inches long, from a single point; the balls being electrified there is a force of repulsion between them so that they separate and remain in equilibrium 4 inches apart; find the force of repulsion and the tension of each thread. (B. of E., I., 1904.)

6. A weight of 1 ton is hung from two hooks 20 ft. apart in a horizontal platform by two chains 15 and 10 ft. long; find by construction and measurement the tension in each chain. (B. of E., II., 1903.)

7. A chain weighing 800 lbs. is hung from its two ends, which are inclined to the horizontal at 40° and 60° respectively. What are the forces in the chain at the points of suspension? (B. of E., A.M. I., 1903.)

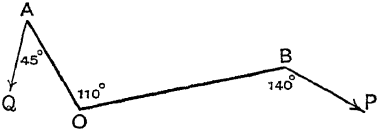

FIG. 160.

8. The figure (Fig. 160) shows a bent lever *AOB* with a frictionless fulcrum *O*. *AO* is 12″, *BO* is 24″. The force *Q* of 1000 lbs. acts at *A*; what force *P* acting at *B* will produce balance? What is the amount and direction of the force acting at *O*? (B. of E., A.M. II., 1904.)

9. A thread is fastened by one end to a fixed point A, and carries at its other end a weight W of 20 lbs. To a point B of the thread a second thread BC is fastened and this second thread is pulled at the end C by a force equal to the weight of 8 lbs.; when the system comes to rest it is found that BC is horizontal. Shew the system when at rest in a diagram drawn to scale, find the angle which AB makes with the horizon and the tension set up in AB. (B. of E., I., 1904.)

10. Draw two lines OA and OB and let $A\hat{O}B$ be an angle of 37°; suppose that R, the resultant of two forces P and Q, is a force of 15 units acting from O to B; suppose also that P is a force of 8 units acting from O to A. Find, by a construction drawn to scale, the line OC along which Q acts, and the number of units of force in Q. (B. of E., I., 1904.)

11. In a common swing gate the weight is borne by the upper hinge. The distance between the upper and lower hinges of such a gate is 3·5 ft. If a boy weighing 119 lbs. gets on the gate at a distance of 8 ft. from the post, find the magnitude and direction of the pressure he exerts on the upper hinge. (B. of E., II., 1905.)

12. A machine of 5 tons in weight is supported by two chains; one of these goes up to an eyebolt in a wall and is inclined 20° to the horizontal; the other goes up to a roof principal and is inclined 73° to the horizontal; find the pulling forces in the chains. (B. of E., A.M. I., 1907.)

13. Fig. 161 shows a weight of 500 lbs. supported by two equally inclined poles. Find the thrust on each pole. (Naval Cadets, 1903.)

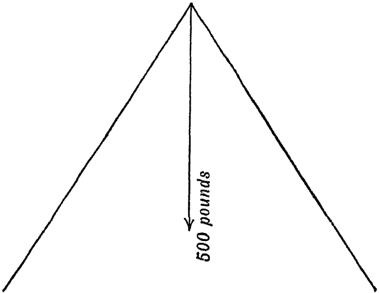

FIG. 161.

14. The bracket shewn in the sketch (Fig. 162) carries a load of 100 kilogrammes at C. Find whether the stresses in AC and BC are thrusts or pulls and the amount. (Military Entrance, 1905.)

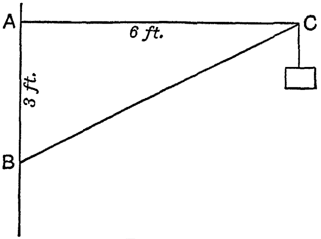

FIG 162.

15. Draw a triangle ABC with AB vertical, and let AC and BC represent two weightless rods, joined together by a smooth hinge at C and fastened by smooth hinges to fixed points at A and B; a weight W is hung from C; shew that one of the bars is in a state of tension, and the other of compression; also shew how to calculate the stresses.

Obtain numerical results in the following case: $AB=4$, $BC=3$, $CA=2$, and $W=18$ tons. (B of E., II., 1905.)

16. In Fig. 163 W is a weight of 170 lbs. hanging from a joint at A by a chain that weighs 20 lbs. The joint is supported by rods AB and AC fixed at B and C. Find the stress in each rod, and say whether it is a thrust or a pull. (Naval Cadets, 1904.)

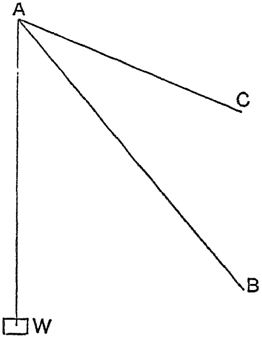

FIG. 163.

17. The two bars AC and BC (Fig. 164), hinged at A and B, and hinged together at C, carry a load of 170 lbs. at C. Find the stress in each bar.

(Inspector of Ordnance Machinery, 1904.)

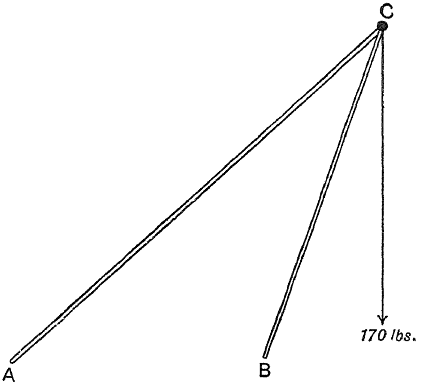

FIG. 164.

18. A body whose mass is 2 cwts. rests on a smooth inclined plane; it is maintained in position by a force of 40 lbs. acting parallel to the surface of the plane, and by a horizontal force of 110 lbs. Determine in any way the angle of inclination of this plane. (Military Entrance, 1905.)

19. A weight slides freely on a cord 2·6 metres long, the ends of which are attached to fixed pegs P and Q; P is 1·4 metres from the vertical through Q and 30 centimetres below the horizontal through Q. Draw to a scale of $\frac{1}{10}$th a diagram shewing the position of equilibrium. Determine the tension in the cord and the proportion of the weight borne by each peg.

Denoting the span or horizontal distance between the pegs by S, the height of one peg above the other by H, and the length of rope by L, find expressions for the horizontal and vertical distances of the weight from the lower peg. (Military Entrance, 1905.)

20. A weight is supported by a tie and a horizontal strut; find how the pull in the tie varies as the inclination changes, and plot a curve giving the pull as a function of the angle of inclination of the tie.

(Military Entrance, 1905.)

21. Fig. 165 represents a vertical section (drawn to the scale of 1 inch to a foot) of the roof of a building, ACB being a window which can turn about a hinge at A and which is opened by means of a rope tied to the end F of a light iron bar CF, which is firmly fixed to the window at C. The rope from F passes over a smooth pulley at K and is fastened to a hook E in the roof. Find the tension in the rope when the position of the window is that indicated in the figure. The weight of the window is 30 lbs. and may be taken as acting at G. (Military Entrance, 1905.)

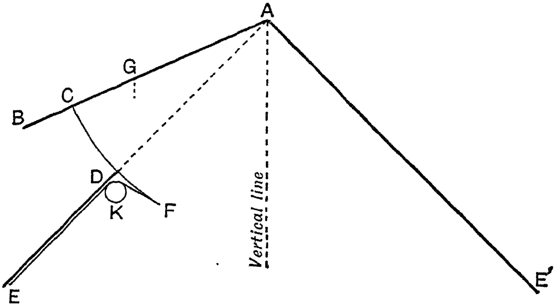

FIG. 165.

22. A uniform bar AB of weight W is freely movable round a smooth horizontal axis fixed at A. It is kept at a fixed inclination i to the horizon by resting against a peg P whose position along the under surface of AB is varied. Represent in a diagram the various magnitudes and directions of the pressures on the peg and the axis A as P is moved along the bar.

(Inter. B.Sc. (Eng.), 1906.)

23. Enunciate the triangle of forces. Shew how to find, by a graphical construction, the angle at which two forces, each equal to 50 lbs. weight, must act on a point that they may have a resultant equal to 75 lbs. weight. (Inter. Sci., 1900.)

24. Draw a triangle ABC with AB vertical and A above B to represent two bars AC and BC freely jointed at C and attached at A and B to points of a wall in the same vertical line. A given weight being suspended from C, determine the natures and magnitudes of the stresses in AC and BC. (Inter. Sci., 1900.)

25. Draw a triangle ABC having the vertical angle A large and the base BC horizontal; produce BC to D. Let AB denote a rod connected by smooth hinges to a fixed point B and to the end A of a rod AC, whose other end C can move in a smooth groove BCD; the weight of the rods being negligible, a force F is applied in the plane ABC at right angles to AB at A; find the force transmitted by the rod AC along the groove. (Inter. Sci., 1900.)

26. A light cord attached to a fixed point O, passes over a fixed pulley Q at the same level as O and at a distance c from it, and supports a weight W attached to its end; another weight w smaller than $2W$ is slung freely over the cord between O and Q; determine the depth below OQ at which this weight will rest in equilibrium. (Inter. Sci., 1900.)

27. Explain a graphical method of finding the resultant of a number of given forces acting on a particle. A light string of length l has its ends fixed at A and B at a *horizontal* distance a apart, and a heavy ring of weight W can slide along the string. Prove that the ring can rest vertically beneath B if a force $W\frac{a}{l}$ be applied parallel to AB. (Inter. Sci., 1904.)

28. A uniform bar AB, 10 ft. long, of weight W is freely movable in a vertical plane, about a smooth axis fixed at A; it is sustained at an angle $\tan^{-1}\frac{3}{4}$ to the horizon by resting against a fixed (smooth) peg at C, where $AC=6$ ft. Find the magnitude and exhibit the lines of action of the pressures at A and C. (Inter. Sci., 1904.)

29. A man stands on a ladder which leans against a vertical wall. Assuming the pressure on the wall to be horizontal, find geometrically the horizontal thrust of the foot of the ladder on the ground. Length of ladder 15 ft., foot of ladder 5 ft. from wall, total weight of man and ladder 3 cwts. acting 5 ft. from the ground (reckoned along the ladder). (Inter. Sci., 1904.)

30. A drawbridge AB, hinged at A (the axis of the hinge being horizontal and perpendicular to AB), is to be raised by a chain attached at B and carried over a pulley C fixed vertically over A at a height $AC=AB$. The resultant weight of the bridge acts through the mid-point of AB. Shew in a diagram how to find the varying tension in the chain due to the weight of the bridge as it is slowly lifted, neglecting the weight of the chain and all friction.

If the bridge weighs 2 cwts., find the tension of the chain and the direction and magnitude of the reaction of the hinge when the bridge is half-open, that is, when AB is at 45° with the horizontal. (Home Civil, I., 1905.)

31. A light bar AB can move freely about the end A, which is fixed, and is supported in a horizontal position by a string CB, C being a fixed point vertically above A. If a weight W be suspended from any point P of the bar, find geometrically the direction and magnitude of the reaction at A and the tension in the stay. $W=10$, $AB=18''$, $AP=12''$, $AC=9''$. (B.Sc., 1904.)

CHAPTER V.

THE LINK POLYGON.

Resultant of three Coplanar Forces (non-parallel).

EXAMPLE. *Draw a triangle* ABC (Fig. 166) *whose sides are* 3, 4, *and* 6 *inches long. Take these as the axes of forces whose magnitudes are* 7, 2, 5 *lbs. weight, and whose senses are given by* AB, BC, *and* CA. *To find the resultant of these forces in magnitude, direction, sense and position.*

Draw the vector polygon for these forces α, β, γ, finding

$$\sigma_1 = \alpha + \beta,$$

$$\sigma = \sigma_1 + \gamma = \alpha + \beta + \gamma.$$

Through B draw BD parallel to σ_1 cutting AC at D. Through D draw DE parallel to σ cutting BC in E.

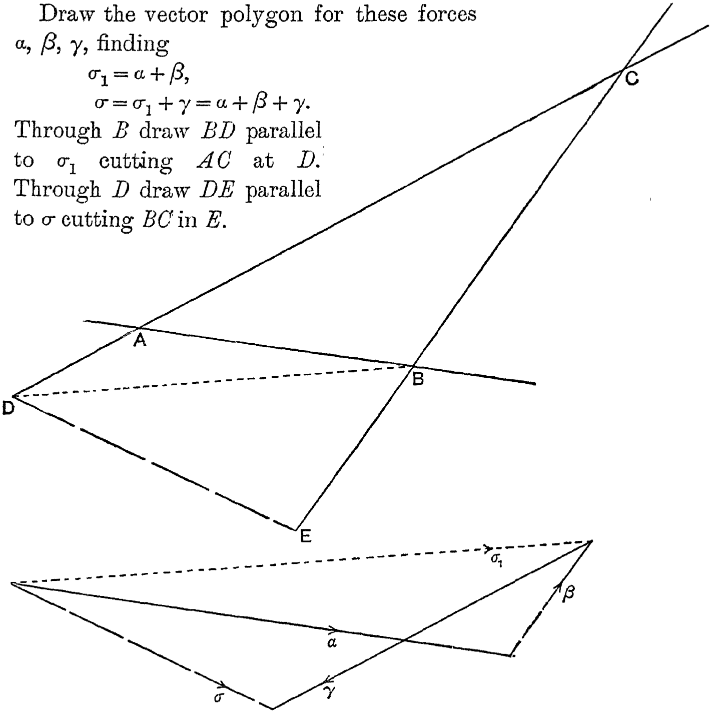

FIG. 166.

Then *DE* is the axis and σ the vector of a force called the *resultant* of the given forces. Measure the magnitude of σ and the angle it makes with *BC*, and the position of *E* with reference to *B* and *C*.

Note that σ is independent of the order of addition of α, β, γ; it is not evident that *E* is independent of the order in which we suppose the forces combined.

(1) Combine the forces in two different orders, viz. (i) α and γ to a resultant through *A* and combine this resultant with β; (ii) β and γ to a resultant through *C* and then combine this with α. Shew in each case that the resultant always cuts *BC* at *E*.

Resultant of any number of Coplanar Forces (non-parallel). When there are more than three forces the process for finding the resultant, if there is one, is simply a continuation of the process explained for three forces, and consists in finding the resultant of two intersecting forces, then the resultant of this and a third force intersecting it, and so on. The whole construction is a repetition of that for two concurrent forces; its validity depends on the truth of the assumption that the order, in which we suppose the forces combined, is immaterial.

(2) Draw an equilateral triangle *ABC* of side 4″; forces of magnitudes 2, 5, 1 lbs. weight act in these sides with senses *AB*, *BC*, *CA*. Find the resultant in magnitude, direction, sense and position.

(3) On squared paper take axes *Ox* and *Oy*. Mark the points whose coordinates are (3, 2), (1, −1), (2, 3) and (−4, 2). Forces of 3, 5, 1·54 and 2 lbs. weight act through these points, their directions and senses being in order.

(i) parallel to *Ox* and in the positive sense.

(ii) making 45°, (iii) making 75°, (iv) making 120° with *Ox* and with senses upwards.

Find the magnitude, direction and sense of the resultant and where it cuts the axis *Ox*.

Resultant by Link Polygon. The construction in the previous examples fails altogether for parallel forces and in many cases involves finding the point of intersection of lines which are nearly parallel.

These difficulties can be overcome by the introduction of two new forces which differ only in sense.

The construction now to be explained depends for its validity on the truth of the suppositions, (i) that such a pair of forces will not affect the equilibrium, (ii) the order of the combination is immaterial.

EXAMPLE. *On any straight line mark four points an inch apart, and draw lines* a, b, c, d *through these as indicated.* α, β, γ, δ *are the vectors of the forces acting in these lines of magnitudes* 1·98, 3·3, 4·15, *and* 2·05 *lbs. weight. Find the resultant.*

Draw the vectors, to the scale 2 cms. to 1 lb. weight, and add them to a resultant vector σ (Fig. 167).

$$\sigma = \alpha + \beta + \gamma + \delta.$$

Mark a point O on the concave side of the vector polygon (called the pole). Join this point to the vertices of the vector polygon $P_1P_2P_3P_4P_5$. The point O should be chosen so that these joining lines are **not** nearly parallel to any of the vectors. For this reason the concave side is the better position for O.

Mark any point A on a and through it draw a line, e, parallel to $\mathbf{OP}_1$. The latter line is a vector, call it ϵ.

Through A draw AB (cutting b at B) parallel to

$$\mathbf{OP}_2 (= \epsilon + \alpha).$$

Through B draw BC (cutting c at C) parallel to

$$\mathbf{OP}_3 (= \epsilon + \alpha + \beta).$$

Through C draw CD (cutting d at D) parallel to

$$\mathbf{OP}_4 (= \epsilon + \alpha + \beta + \gamma).$$

Through D draw DE (cutting e at E) parallel to

$$\mathbf{OP}_5 (= \epsilon + \alpha + \beta + \gamma + \delta).$$

Through E draw r parallel to

$$\mathbf{P}_1\mathbf{P}_5 (= \sigma = \epsilon + \alpha + \rho + \gamma + \delta - \epsilon = \alpha + \rho + \gamma + \delta).$$

Then r is the axis and σ the vector of the resultant of the given forces.

Proof. Let two forces differing only in sense act in e, and suppose their vectors to be ϵ and $-\epsilon$. At A there are two concurrent forces ϵ and α; these are combined to $\epsilon + \alpha$ acting in AB. At B there are two concurrent forces β and $\epsilon + \alpha$; these

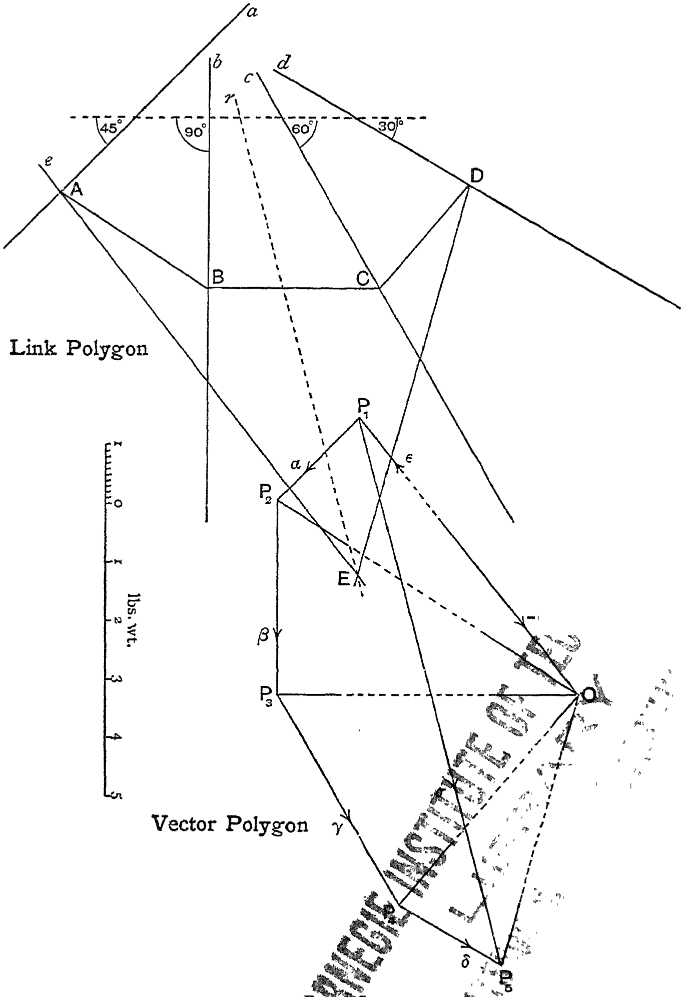
a
b
c
d
r
e
45°
90°
60°
30°
A
B
C
D
E
Link Polygon
P_1
P_2
P_3
P_4
P_5
O
α
β
γ
δ
ε
1
0
1
2
3
4
5
lbs. wt.
Vector Polygon

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FIG. 167.

are combined to $\epsilon+\alpha+\beta$ acting in *BC*. At *C* there are two concurrent forces γ and $\epsilon+\alpha+\beta$; these are combined to $\epsilon+\alpha+\beta+\gamma$

acting in *CD*. At *D* there are two concurrent forces δ and $\epsilon+\alpha+\beta+\gamma$; these are combined to $\epsilon+\alpha+\beta+\gamma+\delta$ acting in *DE*. Finally, at *E* there are two concurrent forces $-\epsilon$ and $\epsilon+\alpha+\beta+\gamma+\delta$; these are combined to a force $\alpha+\beta+\gamma+\delta$ acting in r. Hence r is the axis of the resultant and σ is its vector.

Note that the construction is always possible and never awkward if the pole *O* is chosen properly, for this means that *AB*, *BC*, ... always intersect *b*, *c*, ... at angles never very acute.

(4) Repeat the construction, using a different pole. Is the same axis r obtained?

(5) Repeat the construction, choosing a different point *A* on α.

(6) Repeat the construction, adding the vectors in the order

$$\alpha+\gamma+\delta+\beta.$$

The figure A, B, C, ..., *constructed on the axes* a, b, c, ..., *is called the* **link polygon** (*sometimes the funicular polygon*); *the* **vector polygon** *is often* (*but wrongly*) *called the* **force polygon**.

(7) Find the resultants in Exx. 2 and 3 by the link polygon method.

(8) A wheel has eight tangent spokes placed at equal distances round the hub. The tensions in five consecutive spokes are 3·1, 2·7, 3·3, 1·8 and 2·4 lbs. weight. Find the magnitude, direction, sense and axis of the resultant pull on the hub due to these five spokes, the spokes being tangents to a circle of radius 2·5″.

Equivalent Forces. Any set of forces which would produce the same effect, so far as motion is concerned, as a given system of forces is called equivalent to the latter.

Resultant Force. If a single force would produce the same effect as a given system of forces this equivalent force is called the resultant of the given system.

Not more than one single force can be equivalent to any given set of forces, otherwise forces differing in magnitude or direction or sense or position, or in all together, could produce the same motion in a body. The latter supposition is inadmissible (see Expt. 1, p. 119, and Newton's Second Law of Motion, p. 135).

Equilibrant. Look at the matter a little differently. Suppose a set of forces to be in equilibrium, then any one of the set may be considered as the **equilibrant** of the rest. A force differing only in sense from the equilibrant is the only force that could produce equilibrium with it, and hence is the only single force equivalent to all the rest (Expt. II., p. 119).

Unique Resultant. If, then, a set of forces has a resultant, it can have one only.

That any set of forces has a resultant has not been proved; as a matter of fact, two forces, which differ only in sense and position have no resultant.

The construction given for finding the resultant of any number of coplanar forces consists in finding one after another the single forces equivalent to 2, 3, 4, ... up to the last of the given set, and including in this set two forces differing only in sense. At each step of the process we find the resultant in conformity with experimental results and with Newton's Second Law of Motion. Since there can be one resultant only, the order in which we suppose the forces combined is immaterial.

EXPT. VIII. Punch four holes in a piece of cardboard, and suspend it as in Expt. VI., Chap. IV. Mark the lines of the forces and the corresponding magnitude and sense of each pull.

By vector and link polygons construct the resultant of three of these, and hence shew graphically that this resultant differs only in sense from the fourth force.

Notation (Bow or Henrici). For graphical work it is often (but not always) convenient to have a different notation from that used hitherto. The axis of the force is indicated by two letters (or numbers), one on each side of the force, whilst the vector of the force is indicated by the same two letters in capitals placed at its ends. Thus *ab* (Fig. 168) is the axis, and **AB** the vector, of a force.

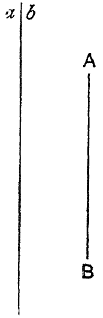

FIG. 168.

EXAMPLE. *Mark five points* P, Q, R, S, T 3 *cms. apart on a straight line, and draw lines through these points at angles* 65°, 90°, 70°, 90° *and* 65°, *as indicated in Fig.* 169. *Letter the spaces between the lines* a, b, c, d, e, f, *as indicated. The forces in these lines are given by the vectors* **AB**, **BC**, **CD**, **DE**, **EF**, *and represent* 3, 2·15, 2·08, 2·7, 2·85 *lbs. weight respectively. Find the resultant force.*

Choose a convenient pole O (Fig. 169). Through the space a draw R_1R parallel to OA; through the space b draw R_1R_2 parallel to OB; through c draw R_2R_3 parallel to OC; through d draw R_3R_4 parallel to OD; through e draw R_4R_5 parallel to OE; and finally through f draw R_5R parallel to OF. The lines through the first and last spaces, *i.e.* R_1R and R_5R, intersect at R, a point on the resultant.

Draw, then, through R a line parallel to AF; it is the axis, and **AF** is the vector of the resultant.

Note that it is not necessary to draw the radial lines OA, OB, OC, ...; in fact it is better not to do so, as the crossing of the lines at O tends to make the exact position of the pole doubtful.

The advantage of the space notation consists in its rendering mechanical the order of drawing the lines. A corresponds to a, B to b, and so on. The more important advantage of uniqueness of construction will be better seen when stress diagrams are under consideration.

In some cases where the axes of the forces cross, some little care is necessary in choosing a good order in which to take the lines and a convenient pole, so that the construction lines may not intersect off the paper.

EXAMPLE. *Draw a parallelogram having adjacent sides of* 2·76 *and* 2·53 *inches, the included angles being* 75° *and* 105°. *Letter the spaces as indicated* (i.e. *taking the parallel sides in order and not the adjacent sides*). *The forces acting in the sides are given by the vectors* **AB**, **BC**, **CD**, **DE**, *and are of magnitudes* 10·3, 4·2, 3·1 *and* 5·6 *kilogrms. weight. Find the resultant.*

Choose a pole O somewhere near the position indicated (Fig. 170)

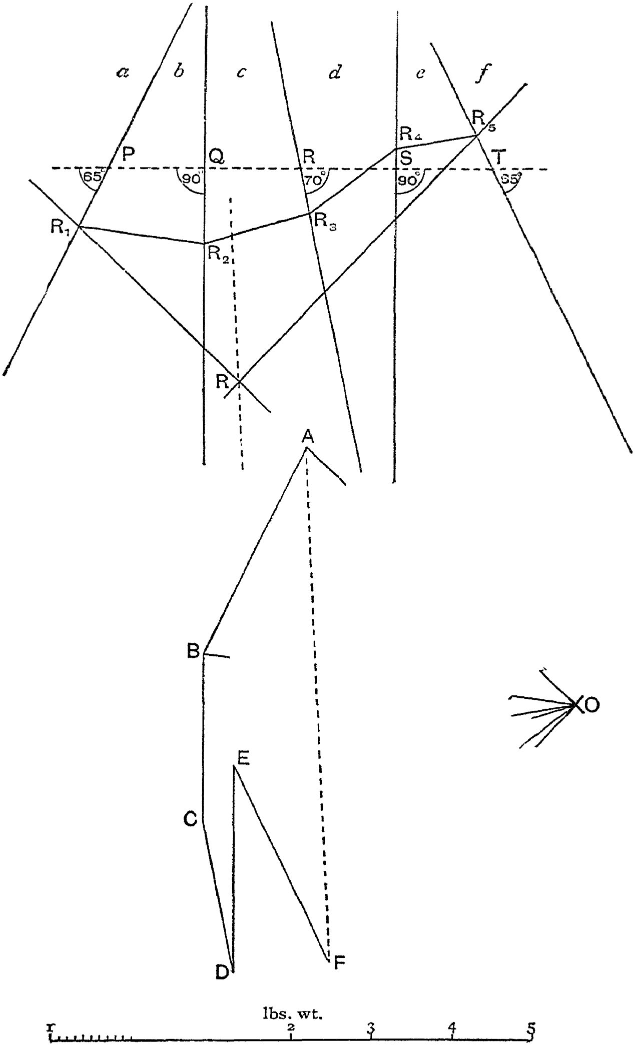
a
b
c
d
e
f
P
Q
R
R₄
S
R₅
T
65°
90°
70°
90°
65°
R₁
R₂
R₃
R
A
B
C
D
E
F
O
lbs. wt.
1
2
3
4
5

FIG. 169.

and construct the link polygon and the axis of the resultant as shewn.

In Fig. 170 the lines from *O* to *A* and *A* to *O* are the arbitrary vectors. The first line in the link polygon is drawn parallel to *OA*, and the second is drawn through the space *b* parallel to *OB*. **AE**, shewn dotted, is the vector of the resultant force; its axis is the dotted line in the link polygon.

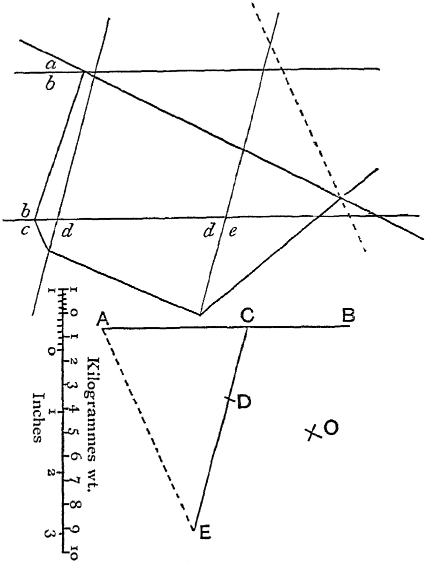

Fig. 170.

(9) Take the order in which the forces are combined differently, *e.g.* take the adjacent sides in order contraclockwise.

(10) Combine the forces directly without using a pole.

Parallel Forces (Like). **Parallel forces** are but a particular case, and the construction for their resultant does not differ in any respect from that given for the general case.

Parallel forces having the same sense are said to be **like**; if they have opposite senses they are **unlike**.

Example. *Fig.* 171 *is a diagram of the four pairs of driving wheels and the trailers of a modern locomotive. The weights borne by the wheels, taken in order from left to right, are* 15, 17, 17, 17 *and* 13 *tons weight, their distances apart are* 6′, 5′ 8″, 6′ and 7′ 6″. *Find the axis of the resultant thrust on the rail.*

Letter the spaces as indicated, and draw the vector and link polygons. Fig. 171 shews *R*, a point on the axis of the resultant, distant 1″ to the right of the centre of the third pair of wheels.

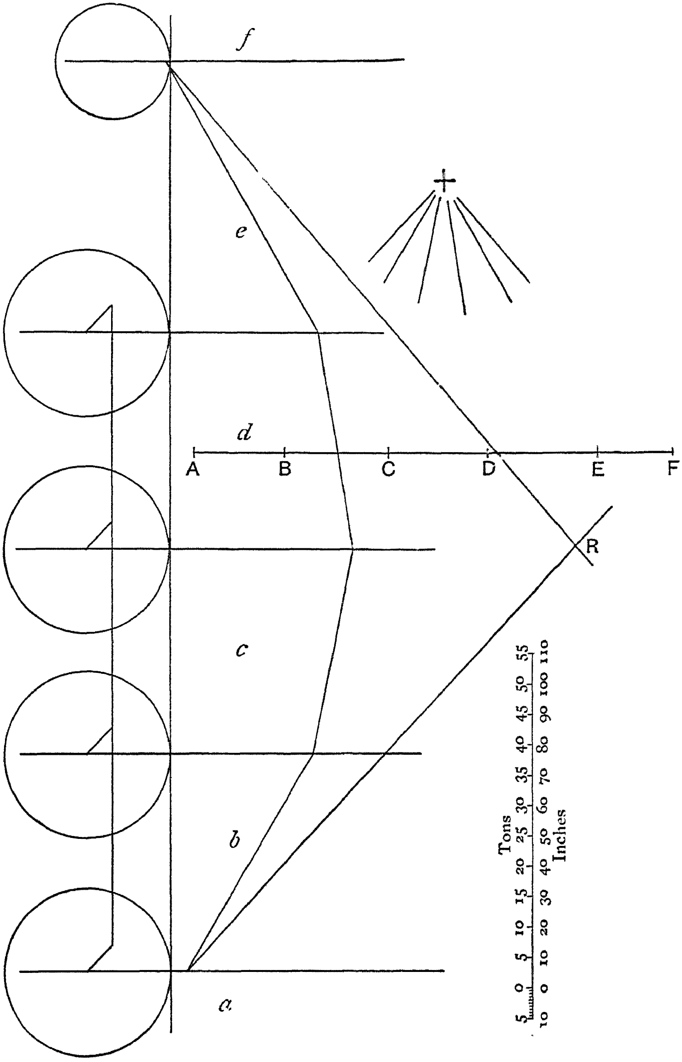
f
e
d
c
b
a
A
B
C
D
E
F
R
Tons
5 0 5 10 15 20 25 30 35 40 45 50 55
Inches
10 0 10 20 30 40 50 60 70 80 90 100 110

FIG. 171.

(11) Find the resultant of two weights of 5 and 7 lbs., distant apart 11″, hung from a horizontal rod.

(12) Find the resultant of six equal weights (1·23 lbs. each) hung from a horizontal rod at distances, from left to right, of 1·7, 1·04, 1·83, 2·02, 0·97 inches apart.

(13) Three men pull at parallel ropes attached to a block in a horizontal plane with forces 51·7, 65·2 and 55·4 lbs. weight; the first two ropes are 2 ft. 8 in. apart; where should the third rope be so that the resultant pull should be in a line midway between the first two ropes.

(14) Three weights W_1, W_2, W_3 are placed in a line on a table, the distance apart of W_1 and W_2 is 2 ft., and of W_2 and W_3 3·2 ft. If $W_1=7$ lbs. and $W_2=4$ lbs., find W_3 if the resultant push is to be midway between W_2 and W_3.

Parallel Forces (Unlike). Should some of the parallel forces be of opposite sense to the rest, the corresponding vectors in the vector polygon must be drawn in their proper sense, the construction is otherwise exactly the same.

EXAMPLE. *Five men pull on a yacht which is stuck on a mud bank by parallel ropes (in the same plane). Find the resultant pull on the yacht, the distances apart of the ropes in ft. and the magnitudes in lbs. weight and senses of the pulls being as given in Fig.* 172.

Set off $AB=5·8$, $BC=7·1$, $CD=7·9$ cms. downwards; and $DE=6·3$ and $EF=5·5$ cms. upwards. The vector sum is **AF** and represents the resultant in magnitude, direction and sense. Draw the link polygon as before, keeping to the order of the letters; finally, P is found as the point of intersection of the first and last lines of the link polygon and therefore is a point on the axis (r) of the resultant.

(15) Find the resultant of two parallel forces 10 and −15 lbs. weight, the axes being 3 ft. apart.

(16) Six parallel forces act on a rod; the magnitudes are 10, −15, 8, −12, 7 and −20 lbs. weight at distances 2, 8, 9, 11, 12, 15 inches from one end; find the resultant force and where its axis cuts the rod.

(17) $PQRS$ is a square of side 3″; a force of 15·3 lbs. weight acts along PQ, one of 8·2 lbs. weight along QR, one of 9·8 lbs. weight along SR and one of 18·4 lbs. weight along SP. Find the resultant in magnitude, direction and sense, and the point where its axis cuts QR.

(18) Find the equilibrant of three parallel forces of magnitudes 8, – 7 and – 2 cwts., the distances apart of their axes being 1 and 1·6 yards.

(19) In a certain locomotive there are four pairs of driving wheels whose distances apart are all 5′ 8″; the distance between the last wheel and the first wheel of a coupled goods truck is 9′ 10″. The truck has three pairs of wheels whose distances apart, from front to rear, are 6′ 3″ and 6′.

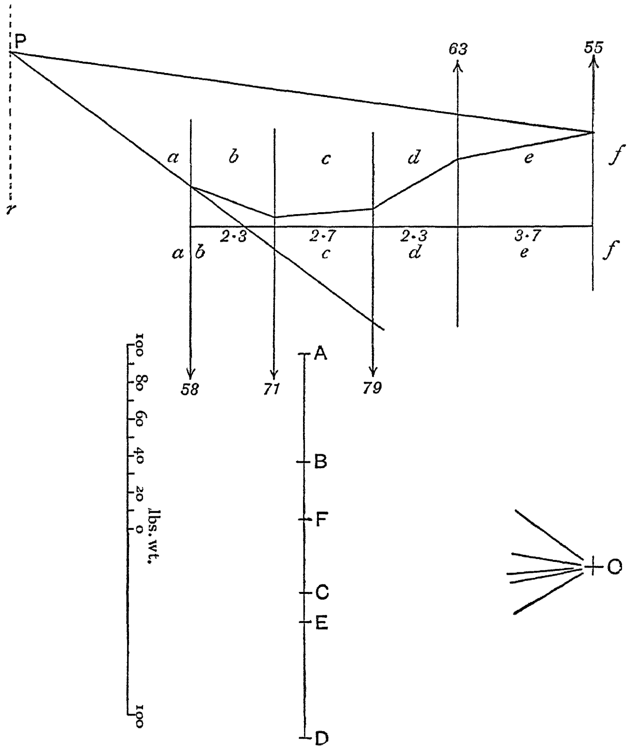

FIG. 172.

The thrusts of the wheels taken in order from the leading driving wheel are 12 tons 10 cwts., 14 tons 8 cwts., 12 tons 14 cwts., 9 tons 13 cwts., 9 tons 12 cwts., 9 tons 15 cwts. and 7 tons 5 cwts. Find the axis of the resultant thrust on the rails.

Vector Polygon Closed. Two Forces. When the vector polygon is closed there is evidently no resultant force, but it does not follow that the forces are in equilibrium.

EXAMPLE. *Draw two parallel lines* ab *and* bc 3 *inches apart, and suppose forces of* 10 *and* -10 *lbs. weight to act in these lines; go through the construction for finding the axis of the resultant.*

Draw the vector polygon (Fig. 173), $AB = 10$ cms. downwards, $BC = 10$ cms. upwards; it is of course closed, since the starting and ending points are the same. Choose a pole O and draw OA, OB, OC. Through space a draw K_1P_1 parallel to OA; through b draw P_1P_2 parallel to OB; and through c draw P_2K_2 parallel to CO.

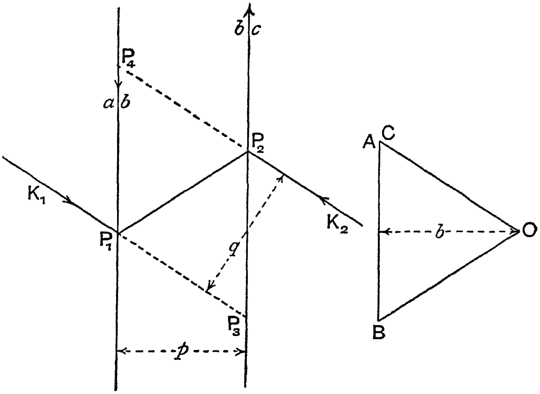

FIG 173.

The theory of the construction is: in K_1P_1 we suppose two forces **OA**, **CO** differing only in sense; **OA** is combined with **AB** to the resultant **OB** acting along P_1P_2; then **OB** is combined with **BC** to the resultant **OC** in P_2K_2; and we have, finally, **CO** in K_1P_1 and $-$**CO** in P_2K_2.

DEFINITION. *Two forces which differ only in position and sense are called a* **couple of forces** *or shortly a* **couple.**

The construction on p. 184 shews that the given couple is equivalent to the final couple, and, since the pole O may be anywhere, there is an infinite number of couples equivalent to any one couple.

To see the connection between the couples, measure the perpendicular distance between the forces. Shew that the product, force × the perpendicular distance between the couple, is the same, *i.e.* shew that $AB \times p = OC \times q$ where p and q, the distances between the axes, are called the arms of the couples.

$AB \, . \, p$ measures the area of a parallelogram whose opposite sides are **AB** and **– AB**, and is called the **momental area of the couple**, if account be taken of the sense of the area.

An area is considered positive if its boundary is given a contraclockwise sense, and negative if the boundary is clockwise. Taking the sense of the momental area as given by the sense of one of the forces we see that the momental area has the same sense in the two cases.

(20) Use in turn four other poles for the vector polygon, taking at least one pole on the opposite side of AB to that in Fig. 173. Calculate in each case the momental area of the equivalent couple, and see that it is equal to $AB \, . \, p$, and of the same sense.

Proof that the construction does give couples of equal momental areas. Produce K_1P_1 to cut bc in P_3, and K_2P_2 to cut ab in P_4. Then $P_1P_3P_2P_4$ is a parallelogram, and $P_1P_2P_3$ and $P_1P_4P_2$ are similar to the vector triangle OAB.

The area of $P_1P_3P_2P_4 = P_1P_3 \cdot q = P_2P_3 \cdot p$;

$$\therefore \frac{p}{q} = \frac{P_1P_3}{P_2P_3} = \frac{OA}{AB};$$

$$\therefore pAB = q \, . \, OA;$$

or, denoting by F and P the magnitudes of the forces represented by **AB** and **AO**

$$pF = qP,$$

i.e. the momental areas of the two couples are equal in magnitude. A simple inspection of Fig. 173 shews that the senses are the same.

Unit of Momental Area. This unit has no special name; if the force be measured in lbs. weight, and the distance in ft., the momental area would be in lbs. ft. (Not ft. lbs.—a term which has a totally different meaning.) The units employed must always be distinctly stated.

Vector Polygon Closed (General Case). The vector polygon being closed, the first and the last lines of the link polygon are of necessity parallel, and the simplest equivalent set of forces is a couple (except in the special case when the first and last links are coincident).

EXAMPLE. *Draw a closed vector polygon* ABCDEA *such that* AB = 2·2″, BC = 1·82″, CD = 2·8″, DE = 1″ *and* EA = 4″ *and* BE = 4·1″ *and* CE = 3·2″.

Draw any line cd (Fig. 174) *parallel to* CD, *and* (*on the left-hand side of the paper*) ab *parallel to* AB *cutting* cd *at* P. *On* cd *mark points* Q, R *and* S *where* PQ = 2·64″, PR = 4·61″ *and* PS = 5·93″. *Through* Q, R *and* S *draw* bc, de *and* ef *parallel to* BC, DE *and* EF *respectively.*

Let the vectors represent in magnitude, direction and sense forces to the scale of 1 *cm. to* 1 *lb., and let the lines drawn through* P, Q, R, S *be their axes.*

Find the equivalent couple to the forces whose vectors are **AB**, **BC**, **CD**, **DE** *and* **EA**, *and whose axes are given.*

Choose some convenient pole O within the vector polygon.

Through any point R_1 in *ab* draw R_1R parallel to OA, and R_1R_2 parallel to OB, and proceed as usual with the link polygon construction until R_5 on *ef* is reached, and R_4R_5 is parallel to OE. Finally, draw R_5R parallel to OA.

The theory of the construction is just as before: in R_1R we suppose two equal and opposite forces **OA** and **AO**; the former we combine with **AB** to a resultant **OB** in R_1R_2; **OB** is combined with **BC** to a resultant **OC** in R_2R_3; **OC** is combined with **CD** to a resultant **OD** in R_3R_4; **OD** is combined with **DE** to a resultant **OE** in R_4R_5; and, finally, **OE** is combined with **EA** to a resultant **OA** in R_5R.

The given set of forces has thus been replaced by a force **AO** or − **OA** in R_1R and **AO** in R_5R, *i.e.* by a couple.

Measure the perpendicular distance between R_1R and R_5R in inches, and multiply the result by the number of lbs. represented by OA (either graphically or by actual multiplication of numbers).

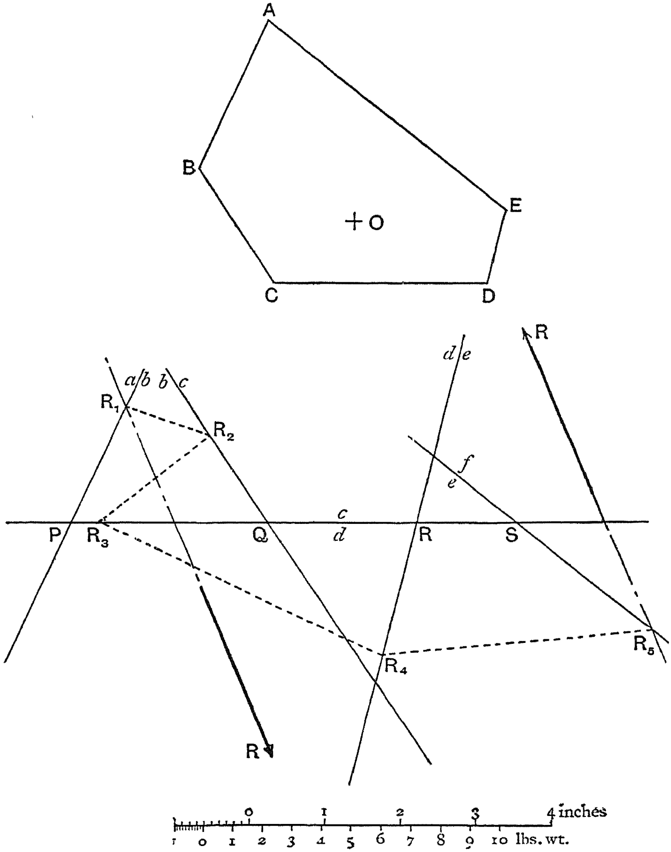
A
B
E
+ O
C
D
R
d/e
a/b
b\c
R1
R2
f
e
c
d
P
R3
Q
R
S
R5
R4
R
0 1 2 3 4 inches
1 0 1 2 3 4 5 6 7 8 9 10 lbs. wt.

FIG. 174.

The product is the momental area of the couple in lbs. and inches. Notice that the sense of the couple is contraclockwise and therefore the sign of the momental area is positive.

(21) With the same vector polygon and axes, take a new pole O_1 outside the vector polygon and shew that the momental area of the couple obtained is the same in magnitude and sign to that obtained previously.

(22) Draw four lines at distances apart of 0·5, 1 and 1·5 inches, and suppose parallel forces of 2·3, 3·7, −1·8 and −4·2 lbs. weight to act in them. Go through the process of finding the resultant and shew that the given set of forces is equivalent to a couple, and find its momental area.

Closed Vector Polygon and Couples. Since the pole **O** may be taken in any position, OA may have any magnitude and direction, and R_1 may be any point on ab; hence the couple equivalent to the given set of forces may have any position in the plane and the forces constituting it may have any magnitude and direction. All the couples found by changing O and R_1 are therefore equivalent, and the connection between the couples is that they all have the same momental area.

Momental Areas are Vector Quantities. Since a momental area has no definite position in space, but is fixed when its magnitude, direction (or aspect of its plane) and sense are given, momental areas are vector quantities.

For coplanar forces the momental areas are all in one plane, and hence they are added by adding their magnitudes algebraically.

Addition of Momental Areas. EXAMPLE. *To find a couple equivalent to three couples having the same sense.*

The magnitudes of the six forces of the three couples are given by lines of length 10·7, 8·65 *and* 12·8 *cms.* (*to a scale of* 1 *kilogrm. weight to an inch*) *the perpendicular distance between the forces constituting the couples are* 12·5, 10·8 *and* 6·25 *cms. The senses of the couples are all clockwise.*

Mark off along any line on a sheet of squared paper (Fig. 175)

$$OA = 12{\cdot}5,\ OB = 10{\cdot}8,\ OC = 6{\cdot}25 \text{ cms.},$$

and on a perpendicular line through O

$$OA_1 = 10{\cdot}7,\ OB_1 = 8{\cdot}65 \text{ and } OC_1 = 12{\cdot}8 \text{ cms.}$$

On the former mark off $OU = 10$ cms.

Draw AA_2 parallel to A_1U, BB_2 parallel to B_1U, and CC_2 parallel to C_1U, cutting the force axis in A_2, B_2 and C_2. Add by the strip method $OC_2 + OB_2 + OA_2$ and scale this with the tenth of an inch scale. It is the momental area of the resultant couple in kilogrms.-cms. (viz. -121 approximately).

Proof. A couple may be supposed to occupy any position in the plane, hence all the couples may be supposed placed so that one force of each lies along OA_1, the other forces will then be parallel to OA_1 and pass through A, B and C respectively.

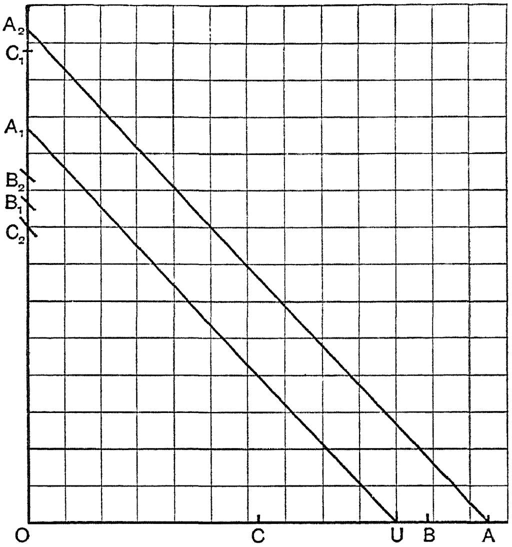

FIG. 175.

Further, a couple may be replaced by any other of equal momental area, hence the couple of force $\mathbf{OA_1}$ and arm OA may be replaced by one of force $\mathbf{OA_2}$ and arm OU. Similarly, the others may be replaced by forces $\mathbf{OB_2}$ and $\mathbf{OC_2}$ and arm OU.

The construction is simply our old construction (p. 46) for reducing an area to unit base (in this case 10 unit base).

Finally, we have forces given by $\mathbf{OA_2}$, $\mathbf{OB_2}$ and $\mathbf{OC_2}$ along OA_1, and parallel forces of opposite sense through U, and the three couples have been replaced by one of force given by

$$\mathbf{OA_2}+\mathbf{OB_2}+\mathbf{OC_2} \text{ and arm } OU \text{ (10 cms.).}$$

Should all the couples not have the same sense, the distances OA_1, OB_1, ... must be set off from O with their proper senses and the corresponding subtraction made by the strip method.

(23) Couples having positive momental areas are given by the annexed table; find the resultant couple

(i) geometrically by reducing each couple to forces distant apart 1″,

(ii) algebraically by adding the momental areas.

Force.	Arm.
23·6	2·84
7·9	4·65
15·4	2·26
10·8	1·92

(24) The couple given in the first and last lines of the above table are negative. Find the resultant couple and its momental area in lb. inches.

Vector and Link Polygons Closed. Refer back to Fig. 174 on p. 187. Imagine R_5R_4 produced to cut R_1R in R_6, then if *ef* be supposed moved parallel to itself to cut R_1R in R_6, R_5R would be the same line as R_1R, and hence the forces **OA** and **AO** would cancel and there would be equilibrium.

EXAMPLE. *Parallel forces act in the lines and have magnitudes and senses as indicated in Fig.* 176. *To show graphically that the forces are in equilibrium* (*approximately*).*

Draw the vector polygon starting with the downward forces **BC, CD, DE**, then **EF** and **FB** upwards (Fig. 176). The upward force **AB** is the same as **FB**. Hence A and F are coincident, and *a* and *f* must be considered the same space. The vector polygon is closed. Choose a convenient pole.

Draw through the space *a* P_5P_4 parallel to OA.
,, ,, ,, *b* P_5P_1 ,, OB.
,, ,, ,, *c* P_1P_2 ,, OC.
,, ,, ,, *d* P_2P_3 ,, OD.
,, ,, ,, *e* P_3P_4 ,, OE.
,, ,, ,, *f* a line ,, OA.

The construction (if properly done) gives the first and last lines identical, viz. P_5P_4 and P_4P_5. But in P_5P_4 acts the force whose vector is **AO**, and in P_4P_5 the force whose vector is $-$**AO**. Two such forces in the same axis must be in equilibrium, and therefore the whole set of forces is in equilibrium. The cancelling of the forces in P_5P_4 is due to the fact that the first and last lines of the link polygon are coincident, *i.e.* the link polygon forms, like the vector polygon, a closed figure.

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* Graphical work will not as a rule give results correct to more than 3 figures, the numbers given in the example are correct to 1 in 1200.

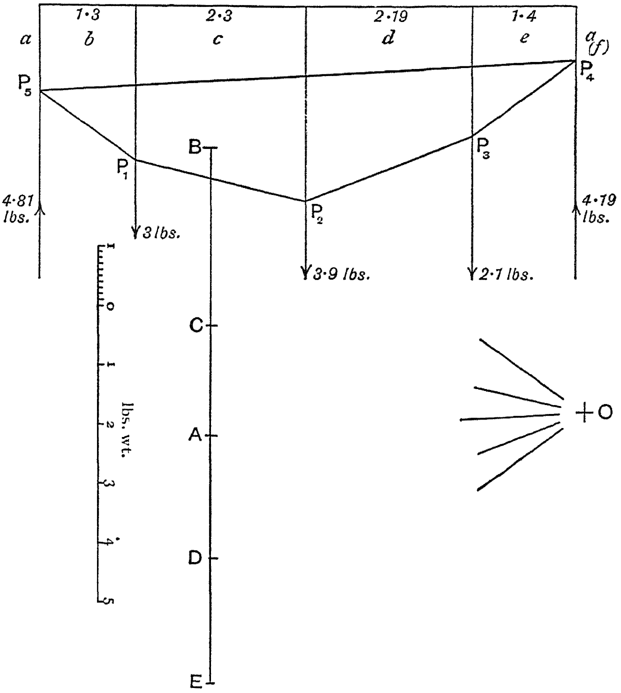

1·3
2·3
2·19
1·4
a
b
c
d
e
a
(f)
P5
P4
B
P1
P3
P2
4·81 lbs.
3 lbs.
3·9 lbs.
2·7 lbs.
4·79 lbs.
1
0
1
2
3
4
5
lbs. wt.
C
A
O
D
E

FIG. 176.

If both the vector and the link polygons for a set of forces are closed, the forces are in equilibrium.

Proof. The general proof of this theorem is seen easily from the construction when the vector polygon is closed, Let $ABCDEA$ (Fig. 177) be the closed vector polygon for certain forces. Then if O be the pole, the first link for the link polygon is parallel to OA, and has, finally, a force whose vector is **AO** acting in it. The last link is also parallel to OA, and has a force whose vector is – **AO** acting in it. These form a couple (in general), but if the first and last links coincide, the forces whose vectors are **AO** and **OA** cancel, and the whole set must be in equilibrium.

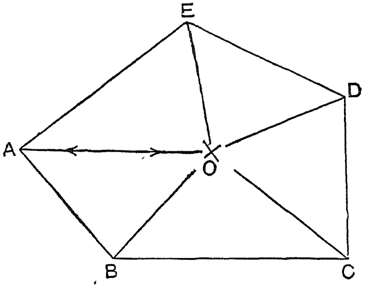


FIG. 177.

EXPT. IX. Punch four holes in an irregular shaped piece of cardboard, and suspend it in front of a drawing board, as in Expt. VI., p. 122.

Mark on the card the lines of actions of the forces, and their magnitudes and senses. Remove the card, and draw the vector and link polygons. Both will be found closed; or as nearly closed as one can expect from the errors incidental to the experiment.

Perform an experiment similar to IX., with five forces.

EXPT. X. Draw a closed four-sided vector polygon on cardboard, the sides being of such lengths that they represent to scale obtainable weights. Draw on the card four non-concurrent lines parallel to these. Fix the card to the drawing board by two pins. Adjust the position of the pulleys so that the corresponding weights may pull on the card along these lines. Remove the pins and see if the card moves.

EXPT. XI. Draw on stiff cardboard a closed four-sided vector polygon $ABCDA$ (Fig. 178), the sides being of convenient lengths to represent to scale obtainable weights.

Draw three non-concurrent lines ab, bc, cd parallel to the corresponding vectors. Take a pole O inside the vector polygon (say the point of intersection of the diagonals).

Mark any point R_1 on ab, and through it draw R_1R_4 parallel to OA; draw R_1R_2 parallel to OB, and cutting bc in R_2; draw R_2R_3 parallel to OC, cutting cd in R_3; then draw R_3R_4 parallel to OD, cutting R_1R_4 in R_4. Through R_4 draw the axis ad parallel to AD.

Then R_3R_4 cuts ad at R_4, and on drawing through R_4 a line parallel to OA we come to R_1R_4 again. Hence the first link R_1R_4, in which lies the force given by AO, coincides with the last link R_1R_4, in which lies the force given by OA. Hence both the vector and the link polygons are closed. Fix the card (with the axes of the forces marked on it) on the drawing board by two stout drawing pins, and adjust the pulleys so that the threads (with their proper weights attached) lie over the axes. Remove the pins, and see that the card does not move.

Devise an experiment for shewing that couples of equal momental areas are equivalent.

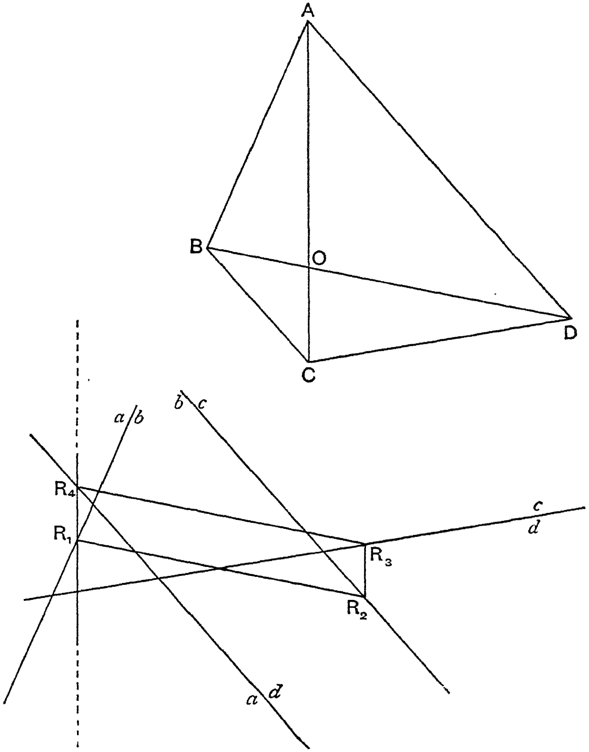


FIG. 178.

Expt. X. shews that in general there is not equilibrium when the vector polygon only is closed.

Expt. XI. shews that there is equilibrium when the link and vector polygons are both closed, and Expt. IX. shews the converse.

Determination of Reactions.

EXAMPLE. *A locomotive has three pairs of driving, one pair of leading, and one pair of trailing wheels, and is stopping on a short bridge of* 40 *ft. span. The centre of the leading wheels is* 6′ 8″ *from one end of the bridge (the left in Fig.* 179) *and the distance between the centres of the wheels are, from the leading to the trailing wheels,* 8′ 9″, 7′, 7′ 9″ and 8′ 3″. *The load each pair of wheels carries is, in the same order,* 9 *tons,* 17 *tons* 13 *cwts.,* 18 *tons* 4 *cwts.,* 18 *tons* 4 *cwts. and* 11 *tons* 9 *cwts. Determine the reactions of the supports (supposed vertical).*

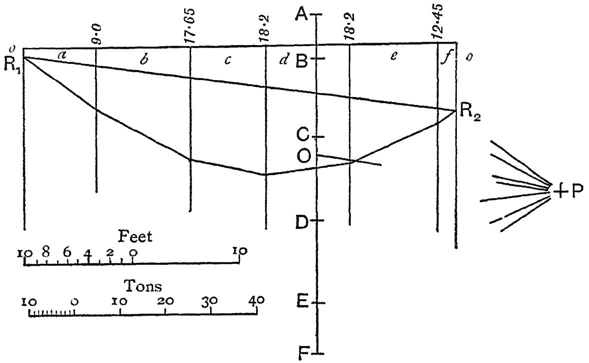


FIG. 179.

Draw the position diagram (Fig. 179) to scale, say 1 cm. to 20 inches, with the reaction and load lines, and letter the spaces *o*, *a*, *b*, *c*, *d*, *e*, *f*, *o* referring to the spaces outside the reaction lines. Draw next the load vectors, say to the scale 0·1 inch to 1 ton, so that **AB** is of length 0·9″, **BC** of length 1·765 inches (*i.e.* nearly 1·77 inches), etc. Choose some convenient pole P and draw through the space a a line parallel to PA cutting the reaction line oa in R_1; through the space b draw a link parallel to PB, and so on to the link through the space f parallel to PF cutting the reaction line fo in R_2. If the reactions in oa and of to maintain equilibrium had been known, the first line of the link polygon would have been through R_1 and the last through R_2, and these would have been coincident and therefore would have been the line R_1R_2 itself.

Hence join R_1R_2, *i.e.* close the link polygon, and through P draw a line parallel to this closing line cutting the load vectors in O; then **FO** is the reaction in *fo* and **OA** that in *oa*, and these are the forces necessary to maintain equilibrium.

Three cases have now been considered:

(i) If a set of forces acts on a body, and the **vector polygon is not closed**, there is a **resultant force** whose vector is the sum of the given vectors and whose line of action is determined by the link polygon.

(ii) If the forces have a **closed** vector polygon they are (in general) equivalent to a **couple** whose momental area can be found from the link polygon.

(iii) If the forces have both the **vector and link polygons closed**, the body is in **equilibrium**.

Incidentally (ii) shewed that all couples which have the same momental area are equivalent.

The third case enables an unknown reaction (or reactions) which keeps a body in equilibrium when under the action of known forces to be found.

(25) A horizontal beam 17 ft. long is loaded with weights distributed as in the Table. The beam being supported on knife edges* at its ends, to find the reaction of these knife edges (neglecting the weight of the beam itself).

Weight in tons, - - -	3·2	3·5	2·1	1·9
Distance from left end in feet,	4·2	7·3	8·6	12·3

(26) A horizontal beam is supported on knife edges at its ends; the length of the beam is 60 ft. and at distances 7, 20, 25, 32, 40 and 49 ft. from the left-hand end are hung weights of 3, 10, 8, 7, 12 and 6 cwts.; find the reactions of the supports.

(27) A beam loaded as in Ex. 26 is supported at the left end and at a point distant 36 ft. from it; find the reactions due to the loads.

---

* The knife edge, shaped like Λ, is simply to ensure that the end is freely supported at one point only or (taking into account the breadth of the beam) on a line perpendicular to the length of the beam.

(28) The beam, loaded as before, is freely supported at a distance of 30 ft. from the left end; can it be supported in equilibrium at the left end, and what is the reaction (given by the polygons) there?

(29) The centre lines of the wheels of a locomotive and tender are from the leading wheel backwards 12′, 9′, 10′ 3·25″, 6′ 10·5″ and 6′ 10·5″ apart; the loads on the wheels are 20 tons 14 cwts., 19 tons 11 cwts., 19 tons 11 cwts., 14 tons 5 cwts., 14 tons 5 cwts. and 14 tons 8·5 cwts. The engine and tender are stopping on a bridge of 60 ft. span and the leading wheel is 7 ft. from one support of the bridge; find the reactions (supposed vertical) of the supports of the bridge.

Non-Parallel Reactions.

EXAMPLE. *A beam is pin-jointed to a supporting wall. Its length is* 25 *feet and it is supported at the other end by a chain of length* 37 *ft. attached to a wall hook* 21 *ft. vertically above the joint. Weights of* 54, 58·5, *and* 45 *lbs. are hung from it at points distant* (*along the beam*) 7, 12, *and* 20 *ft. from the pin. Find the tension of the chain and the reaction of the pin.*

Draw first to scale the position of hook, pin-joint and chain *Z*, *X* and *YZ* (Fig. 180).

Then draw the axes *ab*, *bc* and *cd* of the forces; then the vectors of the forces **AB**, **BC**, **CD** acting in *ab*, *bc*, *cd*, and, finally, the link polygon corresponding to any pole *O*. The resultant of the loads is thus obtained and its axis passes through *R*.

In Fig. 180 the line *ZX* gives the vertical. To obtain the conventional position, the book must be turned round.

Find the point *P* where this axis cuts the chain *ZY*. Join *XP*, and in the vector polygon draw *AE* and *DE* parallel to *XP* and *PY* respectively.

Then **DE** gives the tension in *ZY* (why tension and not compression?) and **EA** the reaction of the hinge.

Since the force whose vector is **AD** and axis *PR* is equivalent to the given three loads, the beam may be supposed to be in equilibrium under the action of this force, the tension in the chain and the reaction at the pin.

These forces must pass through a point, viz. *P*, and then the necessary and sufficient condition for their equilibrium is that their vector polygon should be closed. Hence **DE** must give the force along *YZ*, and **EA** that along *XP*.

Find the vertical reactions through X and Y that would be in equilibrium with the given loads in ab, bc and cd. Project E horizontally to E_1 on AB and see that $\mathbf{DE_1}$ and $\mathbf{E_1A}$ give these vertical reactions.

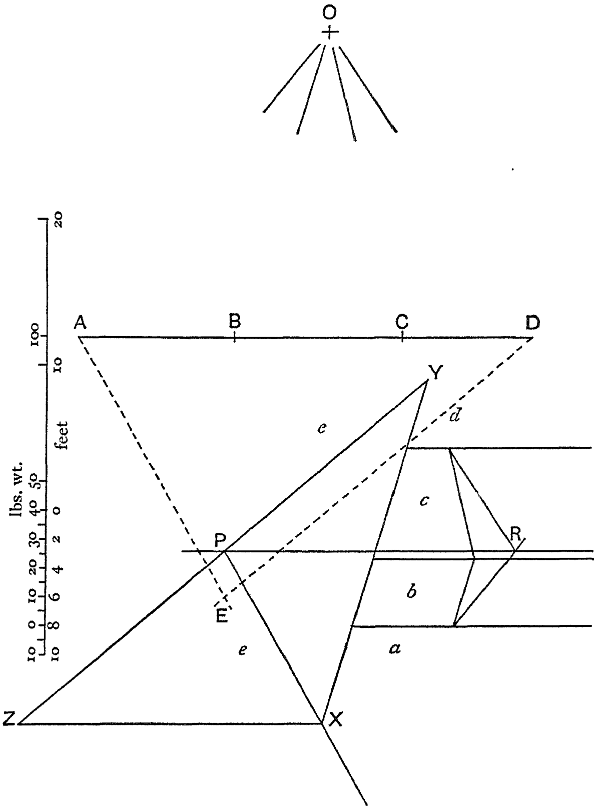

O
A
B
C
D
Y
d
e
c
R
P
b
E
e
a
Z
X
lbs. wt.
100
50
40
30
20
10
0
10
feet
20
10
0
2
4
6
8
10

FIG. 180.

(30) Produce the first and last links to cut XP and PY, and hence shew that the closing line of the link polygon is parallel to OE. Why is this necessarily so?

(31) A horizontal beam 45 ft. long is pin-jointed to a supporting pier at one end (the left), and rests on a smooth horizontal roller at the other. Forces of 12, 8, 20 and 15 cwts. act downwards at points distant 6, 10, 28, and 32 ft. from the pinned end and make angles of 30°, 45°, 90° and 60° with the beam line from left to right. Find the reactions at the ends.

NOTE. The object of the roller is to make the direction of one reaction known, viz. vertical; if neither reaction be known in direction the problem is indeterminate.

First draw the line of the beam and the axes of the known forces, add the vectors of the forces, draw the link polygon and obtain the position of the resultant force of the given set. Mark the point of intersection of the vertical reaction and this resultant, and join this point to the pin. The last line gives the direction of the reaction at the pin. The unknown reaction may then be found from the vector polygon.

(32) Solve the problem in Ex. 31 by projecting the vectors on to a vertical line and drawing the link polygon for these vertical components. Determine thus the reaction of the roller and the vertical component of the pin reaction. The latter combined with the reversed horizontal component of the vectors gives the pin reaction.

(33) A swing gate is hinged at *A* (Fig. 181) to a post and rests against a smooth iron plate at *B*. *AB*=3·5 ft., *CD*=6 ft. and the gate weighs 200 lbs. Supposing the weight of the gate to act at the centre of the rectangle *CD*, find the reactions at the hinge and plate. The distance between *AB* and the gate is 3″.

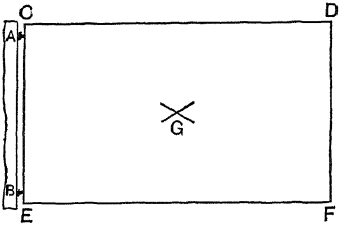


FIG. 181.

(34) A boy of weight 100 lbs. hangs on the gate at *D*. Find the total reactions at *A* and *B*.

(35) The post *AB* is not vertical, but inclined at an angle of 15° to the vertical, so that *CD* slopes (i) downwards, (ii) upwards. Find the reaction when the boy is on the gate in the two cases, if *AC*=*BE*=9″.

(36) In a wall crane *ABC* the beam *BC* is loaded at equal distances as in Fig. 182. Find the tension in the tie rod *AC* and the reaction at *A* and *B*. (The beam *BC* and the tie rod *AC* are pin-jointed at *A*, *B* and *C*.)

(37) *AB* is a uniform beam hinged at *A* (Fig. 183) and weighing 1·6 cwts. It rests on a smooth fixed cylinder *D*, and a load of 0·7 cwt. is suspended from *B*. If *AE* is horizontal and if *AB*=10′, *AC*=7′ and *DE*=2′, find the reactions at *A* and *C*.

(38) A uniform ladder rests against a smooth vertical wall at an angle of 27° with the vertical. The weight of the ladder is 69 lbs. and may be supposed to act at its mid-point. Find the reaction of the ground when a man weighing 12 stones and a boy weighing 7 stones are $\frac{2}{3}$ and $\frac{1}{3}$ up the ladder respectively.

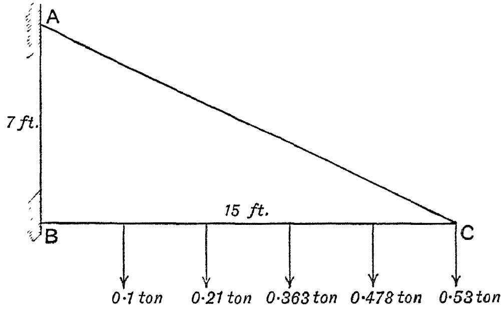


FIG. 182.

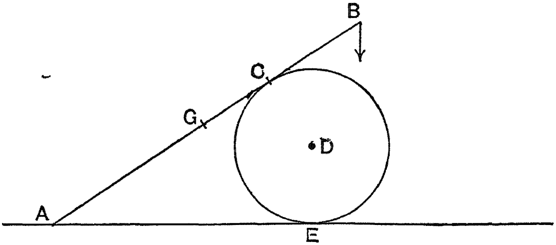


FIG. 183.

Decomposition of Forces. **Any force may be decomposed along any two given axes if they intersect on the axis of the force** (see p. 151). **Any force may be decomposed into two forces parallel to it, having axes in assigned positions.**

Draw any line *bc* (Fig. 184) as the axis of the force, and **BC** its vector. Draw two lines parallel to *bc*, viz. *ab* and *ca*, on opposite sides of *bc*. Choose any pole O, and draw through any point P of *bc*, PP_2 and PP_1 parallel to OC and OB respectively. Join P_1P_2, and in the vector polygon draw OA parallel to P_1P_2. Then **BA** and **AC** are the vectors of the required components.

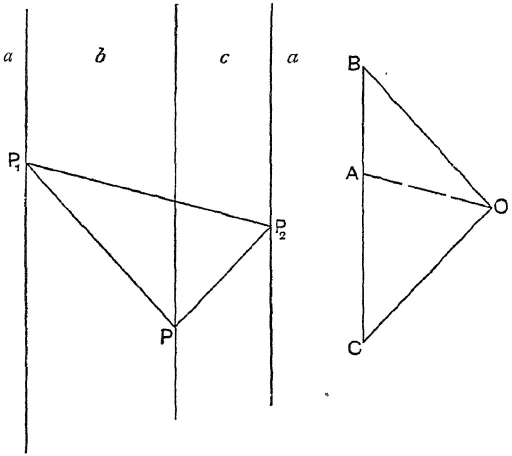


FIG. 184.

Proof. At P, **BC** may be decomposed into two, **BO** and **OC**, acting in PP_1 and PP_2. At P_2, **OC** may be decomposed into two, **OA** in P_1P_2 and **AC** in *ac*. At P_1, **BO** may be decomposed into two, **BA** in *ab* and **AO** in P_1P_2. **OA** and **AO** in P_1P_2 cancel, and we are left with **BA** in *ab* and **AC** in *ca*.

Shortly put, the construction is that for finding the reaction in *ab* and *ca* which will be in equilibrium with the given force in *bc*. These reactions will be the same in magnitude but of opposite sense to the components.

(39) Choose two other poles (one on the side of BC opposite to O) and see that the construction gives the same components.

(40) Decompose a force of given axis and vector into two parallel axes, both axes being on the same side of the force.

(41) Find graphically the components of a force which pass through given points, one direction being fixed.

EXAMPLE. *Find the components, passing through two given points of a force when one of the components has the least possible value.*

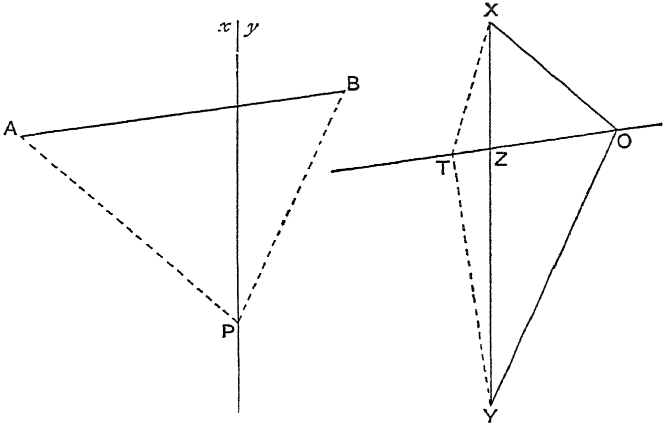


FIG. 185.

Let xy (Fig. 185) be the axis and **XY** the vector of the force, and suppose A and B to be the given points. Join any point P on xy to A and B.

Through X and Y draw XO and YO parallel to PA and PB respectively. Through O, the point of intersection, draw OZ parallel to AB, cutting XY in Z.

If the component through B is to be a minimum, draw YT perpendicular to OZ and join XT; then **XT** and **TY** are the required components.

Proof. To find components through A and B parallel to xy, we may, instead of taking any pole O for the vector polygon, first draw from any point P in xy, PA and PB, and then find the corresponding pole O. The closing line of the link polygon is AB; and hence, on drawing OZ parallel to AB, we get the reactions **YZ** and **ZX** at B and A in equilibrium with **XY** in xy. These reactions must evidently be independent of the pole used to find them, *i.e.* Z is a fixed point on XY. If any other point *P* on *xy* be chosen, then the new pole must be on *ZO*, for *Z* is fixed, and *ZO* is a fixed direction parallel to *AB*.

Further, **XO** and **YO** are two components of **XY** through *A* and *B*, hence the smallest component through *B* will be such that it is perpendicular to *ZO*.

Note that since *ZO* is a fixed line, a simple construction will give the components through *A* and *B* which have any desired relation, say that of equality, or the *B* component twice the *A* component, etc.

(42) Solve graphically the example on p. 201 when *A* and *B* are on the same side of *xy*.

(43) Find components of a given force such that one has an assigned direction and the other is to be as small as possible.

(44) Decompose a given force into two forces equal in magnitude passing forces through points *A* and *B* when *A* and *B* are (i) on the same, (ii) on opposite sides of the given force. (See also Chap. IV., p. 154.) When does the construction fail?

(45) Decompose a given force into two passing through two given points, the magnitudes of the forces having the ratio of *a* to *b*.

(46) A man carries a pole across his shoulder at an angle of 25° with the horizontal. The pole is of length 15 ft., and the distance of the mid-point of the pole from his shoulder is 5 ft. He keeps the pole in position by hard pressure on the front end. In what direction should this pressure be applied so that it may be as small as possible? What direction would make the pressure on his shoulder as small as possible? (Assume that there is sufficient friction at the shoulder to prevent the pole sliding.) Find the pressures in the two cases.

Any force may be decomposed into three forces lying in non-concurrent and non-parallel axes.

Draw any straight line *ab* (Fig. 186) for the axis and a parallel line *AB* for the vector of a force. Draw any three non-concurrent lines *bc*, *cd* and *da* forming a triangle *XYZ*.

Suppose *ab* cuts *XY* in *P*. Then, at *P*, **AB** may be decomposed into two, **AC** and **CB**, acting along *PZ* and *PY*. At *Z*, **AC** may be decomposed into two, **AD** and **DC**, acting along *XZ* and *YZ*.

Hence, **AB** has components **AD**, **DC** and **CB** having *ad*, *dc* and *cb* as axes.

Evidently unless *ab* is parallel to one of the sides of the triangle *XYZ*, we may take as the starting point for the decomposition any of the three points in which *ab* intersects the sides. If the decomposition is unique, the components determined in the three ways should be the same.

The proof that the decomposition is unique will be found in the Chapter on Moments (p. 297).

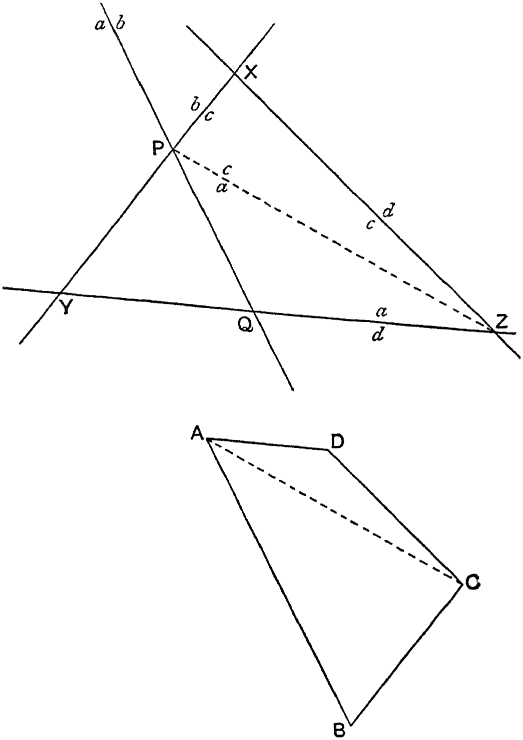


Fig. 186.

(47) Start the decomposition at (i) Q, (ii) R, the points of intersection of ab with YZ and XZ, and shew that the same components are obtained.

(48) Draw any four non-concurrent lines. Assign any value to the force in one, and find the forces in the other three so that the four forces may be in equilibrium.

(49) A weight of 10 tons is suspended from a crane *ABCD* (Fig. 187) at *A*. Find the components along *BC*, *CD* and *DB*; find also the vertical components of *W* through *B* and *C*, and shew that the component along *CD* is the same as that found by the first method.

Also, resolve the *B* component along *BC* and *BD* and compare with previous results.

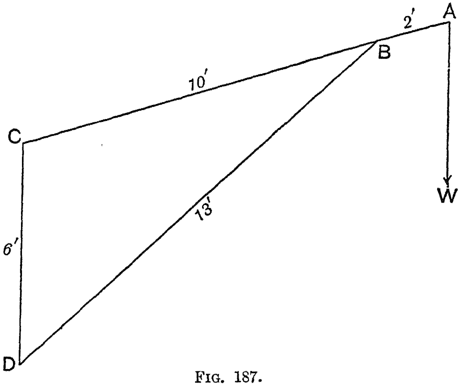


FIG. 187.

(50) *AB*, *BC*, *CA* (Fig. 188) are three light rods pin-jointed together and supported at *B* and *C* in a horizontal line. Find graphically the components along the rods due to a load of 1 cwt. at *D*. Find also the loads at *A* and *C* equivalent to that at *D*, and hence find the components along the rods. Compare the two sets of results.

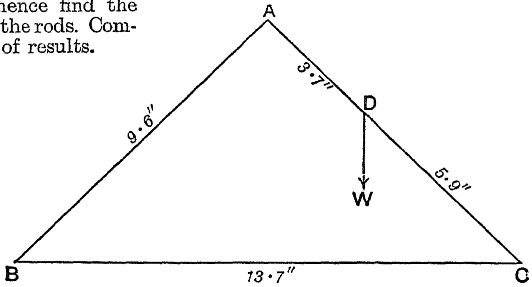


FIG. 188.

(51) *ABC* (Fig. 189) is a wall crane, find the components in *AB*, *BC* and *CA* due to a load of 1 ton applied at *D*; (i) by resolving *W* along *AD* and *CD* and (ii) by finding the parallel components of *W* through *A* and *B*.

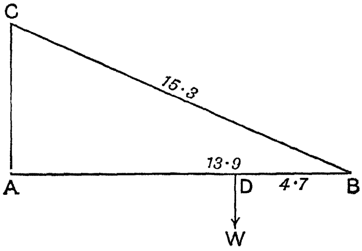


FIG. 189.

(52) In Exercise 51 decompose the load at *D* into equivalent loads at *A* and *B* and find the components of the latter along *BC* and *BA*. Compare with the previous results.

*(53) *AB* and *AC* (Fig. 190) are rafters of a roof; find the total thrust on the walls due to loads of 10 tons at the mid-point of each rafter. (Resolve the load at *M* in directions *MB* and *MC*, and at *C* resolve the latter along *CB* and *CA*. The component along *CB* gives the outward thrust.

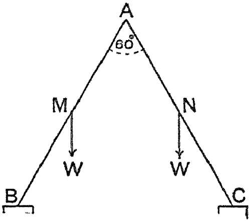


FIG. 190.

EXAMPLE. (*The Toggle Joint.*) AB *is a beam hinged at* A *to fixed masonry.* CD *is a bar pin-jointed to* C (*in* AB) *and to* D. D *is constrained to move along* AD *by means of smooth guides. A force* P *is applied at* B *perpendicular to* AB, *and a force* Q *at* D *along* AD *so that there is equilibrium.* AB = 3·58″, BC = 1·64″, CD = 2·7″ *and* DA = 3·9″. P = 10 *lbs. weight. Find the components of* P *along* AC, CD *and* DA.

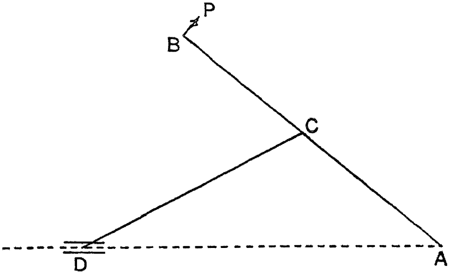


FIG. 191.

The beam *AB* (Fig. 191) is in equilibrium under *P*, a force along *CD*, and the reaction at *A*. These must be concurrent; find the point of concurrence, and hence, in the vector polygon, find the reaction at *A* and shew that this reaction is the equilibrant of the forces at *A* found by resolving *P* along *AC* and *AD* and *CD*.

Find the value of *Q* and the reaction of the guides on the slide at *D*.

(54) Decompose P into parallel forces at C and A. Find the components of the former along CD and CA, and the resultant of the latter and the force along CA. Compare with the former result.

(55) AB (Fig. 192) represents an open French window (plan or trace of on a horizontal plane). It is kept in position by a bar CD, freely jointed at C. The bar has a number of holes in it, any one of which can be fitted over a peg at D so that the angle BAD may have any value from 0 to 120°.

The wind is blowing parallel to AD and would exert a force of 30 lbs. weight if AB were perpendicular to AD. Suppose the resultant force of the wind on the door to act at the mid-point M of AB.

Find the pull in CD, given that $AB = 2' 7''$, $AD = 1' 3{\cdot}5'' = AM$, and $BC = 1$ ft. when

(a) $C\hat{A}D = 20^\circ$,
(b) $C\hat{A}D = 45^\circ$,
(c) $C\hat{A}D = 90^\circ$,
(d) $C\hat{A}D = 120^\circ$.

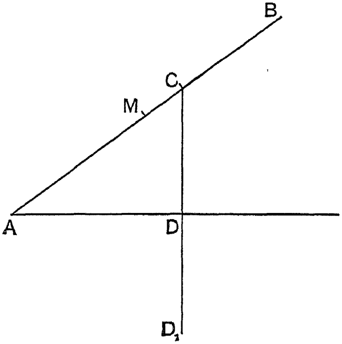


FIG. 192.

GENERAL ANALYTICAL THEORY OF THE COMPOSITION OF COPLANAR FORCES.

***Theorem. Any force is equivalent to a force through any assigned point together with a couple, if the forces have the same vector.**

If a (Fig. 193) is the axis and α the vector of the given force, and O any assigned point, draw through O the axis a_1 parallel to a.

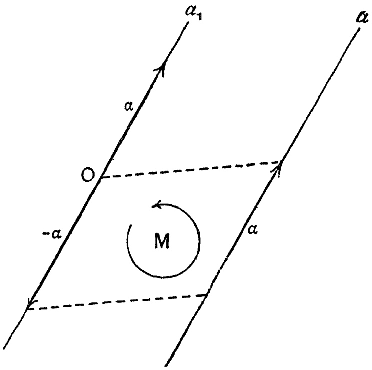


FIG. 193.

Then we may suppose at O in a_1 forces α and $-\alpha$, *i.e.* at O we have a force α which together with the couple α in a and $-\alpha$ in a_1 are equivalent to α in a. *The couple is called the couple of transference.*

***Theorem. Any set of coplanar forces is equivalent to a resultant force through some assigned point and a couple.**

By the previous theorem each force of the set may be replaced by an equal vectored force through O and a couple. The forces, being now concurrent, have in general a resultant found by the vector polygon. The momental areas of the couples may be added to the momental area of a resultant couple, *i.e.* the couples are equivalent to a resultant couple.

***Theorem. Any set of forces reduces to**

(i) a single resultant, or

(ii) a couple, or

(iii) is in equilibrium.

Since a couple may have any position in its plane and may have its forces of any magnitude, provided the momental area is constant, we may replace the resultant couple of the last theorem by an equivalent couple having its forces σ (Fig. 194) and $-\sigma$, where σ is the resultant of the concurrent forces at O.

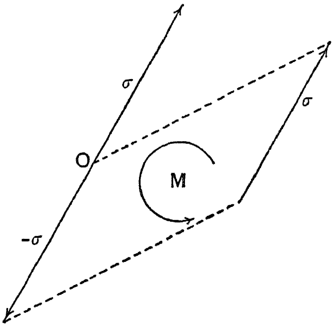


FIG. 194.

If the arm of the couple is p, and S is the magnitude σ, then

$$S . p = M,$$

M being the known momental area.

If, now, the couple be supposed placed so that its force $-\sigma$ passes through O and is in a line with the resultant force σ there, then σ and $-\sigma$ cancel, and we have a single resultant σ at a distance p from O.

Should the resultant of the concurrent forces at O be zero, the set of forces reduces to the couple of momental area M.

If M is also zero, there is equilibrium.

These theorems are only what we had before as direct deductions from the geometrical constructions. The actual determination of the resultant, or the resultant couple, should be effected by the link polygon construction.

MISCELLANEOUS EXAMPLES. V.

1. Draw a triangle *ABC*, having an angle of 45° at *B* and one of 30° at *C*. Let forces of 9, 7 and 4 units act from *A* to *B*, *B* to *C* and *C* to *A* respectively. By construction or otherwise, find their resultant completely, and shew it in the same diagram as the triangle. (B. of E., Stage II.)

2. Draw a square *ABCD* and a diagonal *AC*; forces of 1, 2, 3, 4 units act from *A* to *B*, from *B* to *C*, from *C* to *D* and *D* to *A* respectively; find the sum of their components along *AC*, and also the sum of their components at right angles to *AC*. (B. of E., Stage II.)

3. Draw a triangle *ABC*, and take *D* and *E* the middle points of *BC* and *CA* respectively; if forces *P*, *Q*, *R* act from *A* to *B*, *A* to *C* and *C* to *B* respectively, and are proportional to the lengths of the sides along which they act, shew that their resultant acts from *E* to *D*, and is equal to 2*P*. (B. of E., Stage II.)

4. Define a couple. Explain how to find the resultant of two forces which form a couple and a third force.

Draw a square *ABCD*; a force of eight units acts from *A* to *B* and *C* to *D* respectively; find the resultant. Also find what the resultant would be if the first force acted from *D* to *A*. (B. of E., Stage II.)

5. A horizontal beam 20 ft. long is supported at its ends, loads of 3, 2, 5 and 4 cwts. act at distances 3, 7, 12, 15 ft. from one end. Find, by means of a funicular (link) polygon, the pressures on the two ends. (Inter. Sci. (Eng.), 1904.)

6. Find the resultant of two parallel forces by a graphical construction. Extend this to find the resultant of three or four parallel forces. (Inter. Sci., 1902.)

7. Find the resultant of three parallel like forces of 2, 4 and 3 lbs. weight acting through points in a straight line, distant 1, 3 and 7 ft. from an origin in that line. (Inter. Sci. (Eng.), 1905.)

8. Draw a triangle *ABC*, such that $AB=10$ cms., $BC=14$ and $CA=12$; take B' in *AC*, such that $AB'=3$; and A' in *BC*, such that $BA'=8$. A force of 20 lbs. weight acts in $B'A'$; shew how to replace this force by three forces acting along the sides of a triangle by simple drawing, without using any of the numerical data concerning lengths. (Inter. Sci. (Eng.), 1906.)

9. *ABC* is a right-angled triangle, $AB=12$ and $BC=5$. Forces of 52, 24 and 27 lbs. weight act from *A* to *C*, *B* to *A* and *C* to *B*. Find the resultant of the forces and exhibit its line of action. (Inter. Sci., 1900.)

10. *ABC* is an equilateral triangle, *P* is the foot of the perpendicular from *C* on *AB*. Find in magnitude and line of action the resultant of forces: 10 from *A* to *B*, 8 from *B* to *C*, 12 from *A* to *C* and 6 from *C* to *P*. (B.Sc., 1905.)

11. A locomotive on a bridge of 40 ft. span has the centre line of its leading wheels at a distance of 11 ft. from one abutment, the distance between the centre lines of the wheels are, from the leading wheel backwards towards the far abutment, 9′ 10″, 6′ 8″ and 6′ 8″. Find the pressures on the abutments if the loads on the wheels be 15 tons 10 cwts., 17 tons 10 cwts., 17 tons 10 cwts. and 16 tons 10 cwts.

12. State and prove the rule for finding the resultant of two unlike parallel forces.

Given a force of six units, shew how to resolve it into two unlike parallel forces, of which the greater is ten units; and explain whether the resolution can be made in more ways than one. (B. of E., II., 1904.)

13. Let a horizontal line *AC* represent a rod 12 ft. long, resting on two fixed points *A* and *B*, 10 ft. apart. Each foot of the length of the rod weighs 12 ozs.; a weight of 16 lbs. is hung from *C*. Shew that the rod will stay at rest, and find the pressure at each of the points of support. (B. of E., I., 1904.)

14. *ABC* is an equilateral triangle, and forces P, P_1 and $2P$ act from *B* to *C*, *C* to *A* and *A* to *B* respectively. Find their resultant, and shew, in a carefully-drawn diagram, its direction and line of action. (B. of E., I., 1903.)

15. Forces P, Q and $3R$ act in order along the sides *BC*, *CA*, *AB* of a given triangle. If P, Q, R are proportional to the sides respectively, find completely the force which would balance the three given forces, and shew your result in a carefully-drawn diagram. (B. of E., II., 1907.)

CHAPTER VI.

STRESS DIAGRAMS.

In Chapter IV. the stresses in simple frames, due to loads applied at the joints, were considered. The present chapter is a continuation of the subject. It is shewn that the vector polygon for the forces acting at a point is the stress diagram for the bars meeting there, and that the space notation enables us to draw in one figure—the stress diagram—lines giving the stresses in all the bars of more complicated frames.

Three Bar Equilateral Frame.

Example. *Three bars, each* 3 *ft. long, are pin-jointed together to form an equilateral triangle. The frame is suspended by one vertex, and* 5·5 *lb. weights are hung from the others; to determine the stresses in the bars due to the weights.*

Draw the frame PQR (Fig. 195) to scale (say 1″ to 1′) and letter the spaces as indicated.

Draw the vectors of the two external forces at R and Q (scale say 2 cms. to 1 lb. weight); then, without drawing the link polygon, notice that the upward reaction **CA** at P (= 11 lbs. weight) will keep the frame in equilibrium.

Draw AD and BD parallel to ad, bd, then CD is parallel to cd; and AD, DB, and DC measure the magnitudes of the stresses in the corresponding bars. Scale these lines and tabulate the stresses.

Bar, - - -	*ad* -	*bd*	*cd*
Stress in lbs. wt.,	6·35	3·18	6·35

At the point P of the frame act three forces, viz. the reaction **CA** vertically upwards, and the pushes or pulls of the bars *ad* and *cd*. Hence $CADC$ is the vector polygon for P. The sense of the force at P due to *cd* is given by **DC**, and hence the bar must

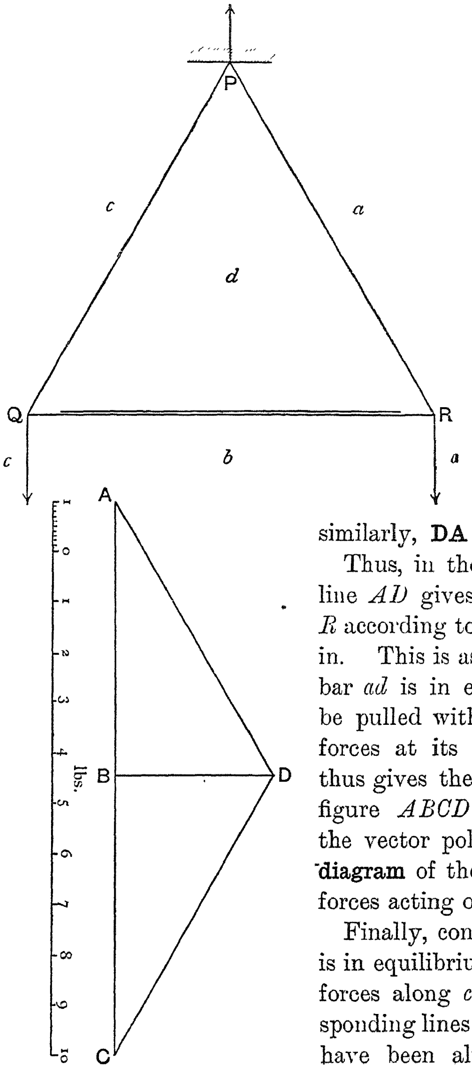


Fig. 195.

pull at *P*. Similarly, the sense of the force in *ad* is given by **AD**, and hence this bar also pulls at *P*.

Now consider the equilibrium of *R*. The forces there are the weight of 5·5 lbs. (**AB**) downwards and the forces due to *ad* and *bd*. The vector polygon is *ABDA*, and, since **AB** is downwards, the force in *bd* is given by **BD** and pushes at *R*; similarly, **DA** pulls at *R*.

Thus, in the vector polygon the line *AD* gives the pull at *P* or at *R* according to the sense it is taken in. This is as it should be, for the bar *ad* is in equilibrium and must be pulled with equal and opposite forces at its ends. The line *AD* thus gives the stress in *ad*, and the figure *ABCD* is called now, not the vector polygon, but the **stress diagram** of the frame *PQR* and the forces acting on it.

Finally, consider the point *Q*. It is in equilibrium under **BC**, and the forces along *cd* and *db*; the corresponding lines in the stress diagram, have been already drawn, and it only remains to determine the senses of the forces at Q. The vector polygon for Q is $BCDB$, hence a force in *cd* (**CD**) pulls at Q whilst **DB** pushes. Hence *bd* pushes at its two ends, Q and R, and is therefore in compression, whilst PQ and RP pull at both ends, and are therefore in tension. As vectors, therefore, the lines AD, DC and DB should have double arrow heads; this may easily lead to confusion, and it is, therefore, better to avoid them altogether and to indicate in the frame those bars which are in compression by drawing fine lines parallel to them. Mark the bar *bd* as being in compression.

Change of Shape in Frames under Forces. The simple triangular frame, just considered, was treated as a rigid body; this was justifiable, since, although some bars may elongate a little and others contract, yet the bars will always adjust their positions to form a closed triangle, and when once the deformation has taken place, the parts retain their relative positions unaltered. That is, for the given forces the frame—after the elongations, etc., have taken place—is like a rigid body. In nearly all practical cases the elongations, etc., are so small that the frame may, so far as change of form is concerned, be regarded as unaltered.

Only frames which have just a sufficient number of bars or strings to keep them rigid under the given applied forces will be considered, and of these only simple cases which can be solved directly by vector polygons will be taken.

The weight of the bars will be neglected unless expressly included in the problem.

EXAMPLE. *Three bars,* PQ, QR *and* RP, *of lengths* 4, 7, *and* 6 *ft., are freely pin-jointed together to form a triangular frame and the frame is suspended by* P. *Weights of* 4 *and* 6 *lbs. are suspended from* Q *and* R. *Draw the frame in its position of equilibrium and find the stresses in the bars.*

The resultant of the weights 4 and 6 at Q and R (Fig. 196) must act through a point S such that $\frac{QS}{SR}=\frac{3}{2}$; hence, by construction, find the position of S in QR.

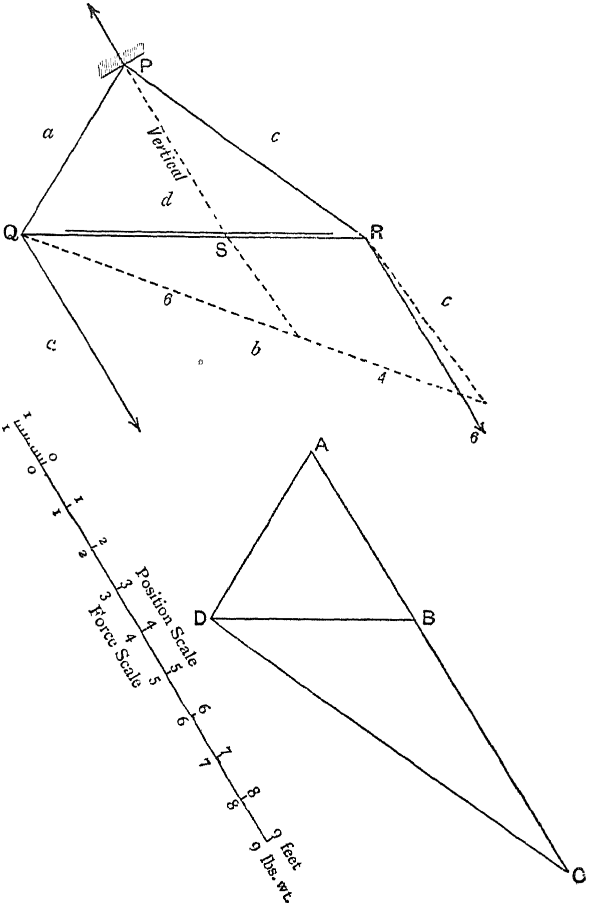

P
a
c
Vertical
d
Q
R
S
6
c
b
c
4
6
A
D
B
C
Position Scale
Force Scale
1 0 1 2 3 4 5 6 7 8 9 feet
1 0 1 2 3 4 5 6 7 8 9 lbs. wt.

FIG. 196.

Then, if *PS* be regarded as vertical, the angles the sides make with the vertical or horizontal can be measured. (If the conventional position be desired, the length *PS* must be set off vertically and then the triangle drawn in.)

Letter the spaces as in Fig. 196 and draw the vector polygon *ABD* for *Q*.

Draw next the vectors for the forces at *R*; the vector triangle is *DBC* where *DC* is parallel to *dc*. *ADC* is then the vector triangle for the point *P*. See that *CA* gives 10 lbs., the reaction at *P*, and that the senses of the forces at the joints are consistent with one another.

Measure the stresses and see which bars are in compression and which in tension, and tabulate the results. Measure the angle *QR* makes with the vertical.

(1) Determine the position of *QR* (Fig. 196) and the stresses in the bars if the weights at *Q* and *R* are 7 and 4 lbs.

(2) An equilateral framework *PQR* of three bars is suspended by means of two vertical strings attached to *P* and *Q* so that *PQ* is horizontal. A load of 17 lbs. is suspended from *R*. Find the stresses in the bars of the frame, stating which are in compression and which in tension.

(3) The equilateral frame of Ex. 2 being suspended from a hook at *P* and a vertical string at *Q*, so that *PQ* makes 15° with the horizontal, determine, by the link polygon, the reaction at *P*, the tension in the string, and find the stresses in the bars. Determine also the reactions at *P* and *Q* from the stress diagram.

Braced Quadrilateral Frame.

EXAMPLE. *Draw to scale the frame* PQRS (Fig. 197), *supported at* Q *and* R *in the same horizontal line, given that* $PQ = 1 \cdot 5$, $PS = 2$, $QR = 3 \cdot 7$ *metres, and* PS *is parallel to* QR *and* $P\hat{Q}R = 60°$. *A load of* 12·1 *lbs. is placed at* P; *determine the stresses in the bars due to this load.*

Letter the spaces as indicated and draw in the stress diagram $AB = 12 \cdot 1$ cms.; then *BD* and *AD* are parallel to *bd* and *ad*. *ABDA* is the vector polygon for *P* and gives the stresses in the bars *bd* and *ad*. **BD** evidently pushes at *P*, so that, since *bd* is in equilibrium, it must push at both ends and be in compression. Similarly, **DA** pushes at *P*, and hence *ad* must also be in compression. At *S* we know the force **AD** (pushing at *S*), hence we can find the stresses in *dc* and *ca* which will give equilibrium. Draw *DC* and *AC*, parallel to *dc* and *ac*, intersecting at *C*; then *ADCA* is the vector polygon for *S*, in which we know the sense **AD** of the force in *ad* at *S*.

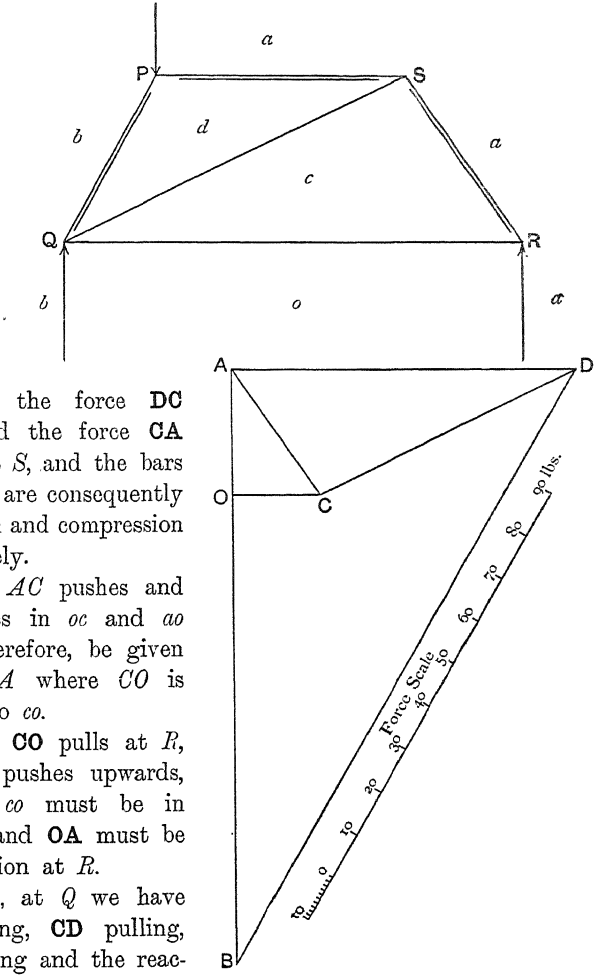

a
P
S
b
d
a
c
Q
R
b
o
a
A
D
O
C
B
Force Scale
90 lbs.
80
70
60
50
40
30
20
10
0
10

FIG. 197.

Hence, the force **DC** pulls, and the force **CA** pushes at *S*, and the bars *dc* and *ca* are consequently in tension and compression respectively.

At *R*, *AC* pushes and the forces in *oc* and *ao* must, therefore, be given by *ACOA* where *CO* is parallel to *co*.

Hence, **CO** pulls at *R*, and **OA** pushes upwards, so that *co* must be in tension, and **OA** must be the reaction at *R*.

Finally, at *Q* we have **CO** pulling, **CD** pulling, **DB** pushing and the reaction, which must therefore, be **BO**.

Notice that given the reaction **BO**, the senses of all the other forces necessary to produce equilibrium are consistent with the senses previously obtained.

Scale all the lines in the stress diagram and tabulate the stresses.

ac	*ad*	*bd*	*dc*	*co*	Bar.
					Stress in lbs. wt.

In the frame diagram mark those bars which are in compression.

The reactions **BO** and **OA** have been obtained, without drawing the link polygon, on the supposition that there is equilibrium.

(4) Find the reactions in *ob* and *oa* due to the load in *ab* by the link polygon, *i.e.* find the components of **AB** in two parallel lines through *Q* and *R*, and compare with the previous results.

Cantilever.

EXAMPLE. PQRST (Fig. 198) *represents a cantilever pin-jointed to a vertical wall at* R *and* S. RS = 5, ST = 2·25, SP = 8·25, PQ = 5·88, QR = 4·4 *ft. The loads at* T *and* P *are* 2300 *and* 3400 *lbs. weight respectively. Find the stresses in the bars.*

Draw the load vectors **AB** and **BC**. Then, since there is equilibrium at *P*, draw *BD* and *CD* parallel to *bd* and *cd*. *BCDB* gives *cd* in tension and *db* in compression.

For *Q*, draw *CE* and *DE* parallel to *ce* and *de*. Then, from *CDEC*, we know that **DC** gives the sense of the force in *dc* on *Q*, and hence *DCED* is the correct sense of the diagram, and *ce* is in tension and *ed* is in compression.

Finally, for *T* draw *AF* and *EF* parallel to *af* and *ef*. We may determine the senses of the forces at *T* either from knowing that in *ab* or that in *ed*. The vector polygon is *ABDEFA* in the sense given by the letters, and hence *ef* is in tension and *fa* in compression.

Indicate on the frame figure the bars in compression, measure the stresses from the stress diagram and tabulate the results.

Bar, - - -	*cd*	*bd*	*ce*	*ef*	*fa*
Stress in lbs. wt.,					

(5) Four equal bars are pin-jointed together to form a square and a fifth bar is introduced diagonally. The frame is suspended by a vertex so that the diagonal bar is horizontal, and the three remaining vertices are loaded with 7 lbs. weight each. Find the stresses in all the bars.

(6) As in previous exercise only the diagonal bar is vertical.

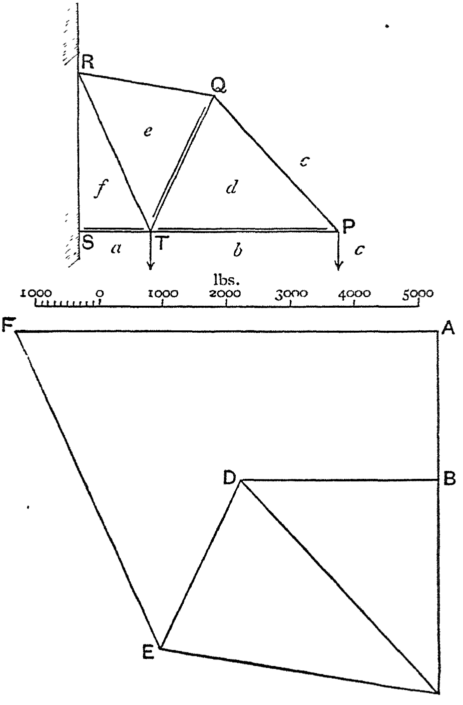


FIG. 198.

*(7) If the weights at the two vertices, where three bars meet, be 4 and 8 lbs., and a third vertex sustain 7 lbs., find the position of equilibrium when the frame is suspended by the remaining vertex. Find also the stresses in the bars. (Draw the frame with the diagonal bar, find the M.C. of the three masses. Join the M.C. to the fourth vertex, this line relatively to the frame is the vertical line. The formal proof for the Centre of Parallel Forces is on p. 298.)

*(8) Five bars of lengths 1·2, 1 2, 1·6, 1·6 and 2 ft. respectively when jointed together form a rectangle *PQRS* and one diagonal *QS*. The frame is suspended by *P* and loaded at *Q*, *R*, *S* with weights of 7, 5 and 11 lbs. Find the position of the frame and the stresses in the bars.

(9) Three rods *AB*, *BC*, *CA* of lengths 7, 6·2 and 5·8 ft. respectively are pin-jointed together. *A* is fixed and *B* rests on a smooth horizontal plane so that *A* is 2 ft. above *B*, and a load of 70 lbs. weight is hung from *C*. Determine the stresses in the bars and the reactions of the supports on *A* and *B*.

(10) The frame *PQRS* (Fig. 199) is loaded at *Q* and *R* with 100 lbs. weights and supported at *P* and *S*. If $PS=15$, $QR=8$ and $PQ=SR=6·6$, find the stresses in the bars.

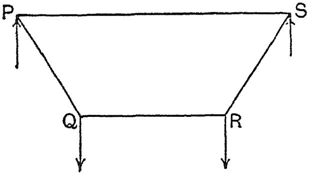


FIG. 199.

Bridge Girder.

EXAMPLE. PQRSTUV (Fig. 200) *represents a short* **N** *girder, the bars forming right-angled isosceles triangles. It is freely supported at* P *and* T *and loaded at each of the joints* Q, R, *and* S *with* 1·5 *tons. Determine the reactions at* P *and* T *and the stresses in the bars.*

The loading being symmetrical, the reactions at *P* and *T* must each be equal to 2·25 tons and there is no occasion to draw the link polygon.

Letter the spaces as indicated and draw the vectors **AB**, **BC** and **CD** of the loads; then, bisecting *AD* at *O*, **DO** and **OA** must be the reactions at *T* and *P*. At *P* the known force **OA** acts and the forces in *aj* and *oj*; these being in equilibrium draw *AJ* parallel to *aj* and *OJ* parallel to *oj*; then *OAJ* is the vector triangle for *P*. The force **AJ** pushes *P*, and **JO** pulls, hence *aj* is in compression and *oj* in tension.

At *Q* there are three bars with unknown forces, and as resolution into three concurrent straight lines is not unique, we must try some other point *U*. At *U* one known force **OJ** acts and two unknowns; draw, then, in the stress diagram *OI* parallel to *oi* and *JI* parallel to *ji*; then *OJI* gives the stresses in the bars meeting at *U*. Also **OJ** gives the sense of the force at *U* along *oj*; **JI** pushes at *U* and **IO** pulls, hence *ji* is in compression and *io* is in tension.

Now return to *Q*, where there are only two unknowns remaining, and draw the vector polygon *ABHIJA* (most of it is already drawn, *IH* and *BH* being the only two lines necessary). See from the sense of this polygon that *bh* and *ji* must be in compression and *ih* in tension.

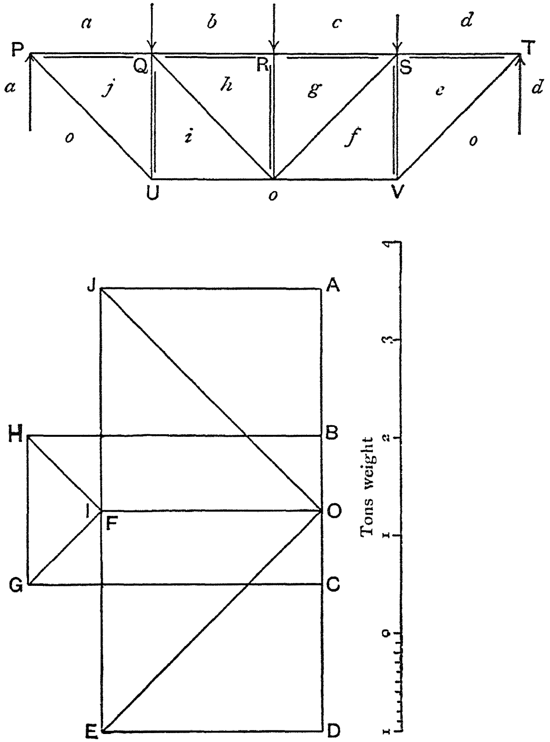

a
b
c
d
P
Q
R
S
T
a
j
h
g
e
d
o
i
f
o
U
o
V
J
A
H
B
I
F
O
G
C
E
D
4
3
2
1
0
1
Tons weight

FIG. 200.

Draw the rest of the stress diagram, noticing, from the symmetry of the loads and the frame, that the stress diagram must also be symmetrical. In consequence of this symmetry the points *I* and *F* of the stress diagram are coincident.

However many bars the frame contains, the method of solution always follows the same lines. The stress diagram is started by drawing the vector triangle for three concurrent forces, of which one is known and the directions of the other two are given by bars in the frame. Sometimes this start can be made at once, but more generally the reactions at the points of support have first to be determined, either by drawing the link polygon for the external forces or by taking moments. For the latter method see Chap. VIII.

Roof Truss.

EXAMPLE. *The frame diagram shewn* (Fig. 201) *represents a bowstring roof truss supported freely at* P *and* T *in a horizontal line;* PQ = 27′, QR = 19′, PU = 19·4′, RV = 11·4′ *and* PT = 63′, *and the angle* UPT = 44°. *The loads at* Q, R *and* S *are* 1·5, 2 *and* 2·5 *tons; determine the reactions at the supports and the stresses in the bars.*

Draw the frame to scale. Set off the vectors of the loads **AB**, **BC**, **CD**, and number the spaces of the frame as indicated. Choose any pole O_1 and through Q_1 any point on the vertical QQ_1, draw Q_1P_1 parallel to O_1A, cutting the reaction line in P_1. Then draw Q_1R_1 parallel to O_1B, R_1S_1 parallel to O_1C, and S_1T_1 parallel to O_1D cutting the reaction line at T_1. Join P_1T_1, thus closing the link polygon, and draw O_1O in the vector polygon parallel to P_1T_1. The reactions at P and T are, therefore, **OA** and **DO.**

Now draw the stress diagram, starting with the vector triangle for the forces at P. The vectors are **OA**, **AJ** and **JO**, shewing that the bar *aj* is in compression and *jo* in tension.

At the point Q are three unknown forces; and since the known forces in *aj* and *ab* cannot be decomposed in three directions, nothing can be done there at present. But at U we have only two unknowns; hence draw OI parallel to *oi* and JI parallel to *ji* to determine the point I; and the stresses in *oi* and *ij* are found. The point Q may now be attacked, for there are now only two unknown forces there.

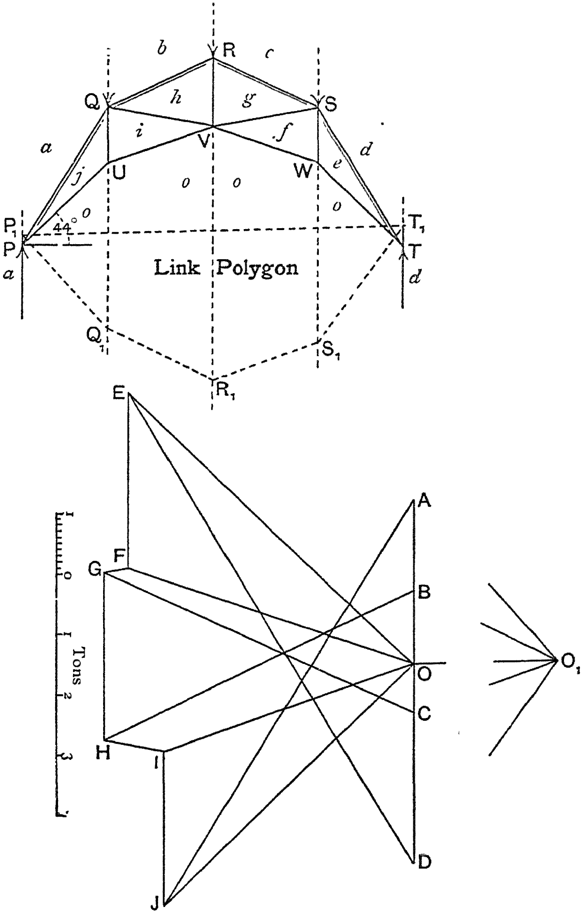

b
R
c
Q
h
g
S
a
i
V
f
d
j
U
W
e
o
o
o
o
o
P₁
44°
T₁
P
T
a
Link Polygon
d
Q₁
S₁
R₁
E
A
F
G
B
O
O₁
C
H
I
D
J
Tons
1
0
1
2
3

FIG. 201.

Complete the stress diagram, mark the bars which are in compression, scale the lines in the stress diagrams and make a table of the stresses as before.

(11) The frame *PQRST* (Fig. 202) is loaded at *T* and *S* with 1 ton weights and supported at *P* and *R*. Find the stresses in the bars if

$$PT = TS = SR = 9 \text{ ft.}$$

and $$RPQ = 30°.$$

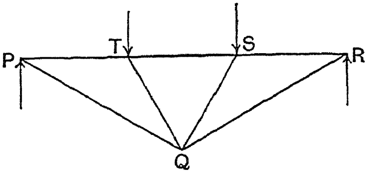


FIG. 202.

(12) *PQRST* (Fig. 203) is a short Warren girder consisting of three equilateral triangles. A load of 4 tons is suspended from *R* and the girder is supported at *Q* and *S*. Find all the stresses.

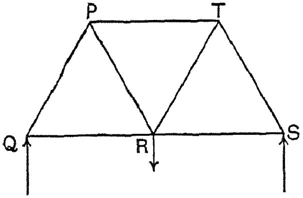


FIG. 203.

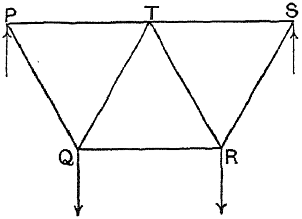


FIG. 204.

(13) *PQRST* (Fig. 204) is a short Warren girder of three equilateral triangles supported at *P* and *S*, and with loads of 2 tons at *Q* and *R*. Find the reactions of *P* and *S*, and the stresses in the bars.

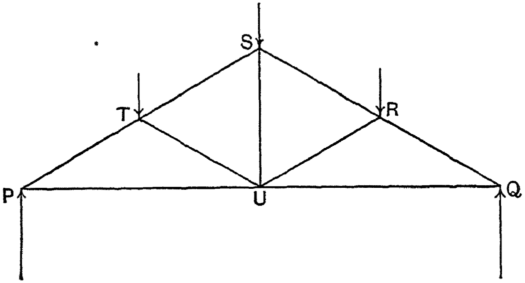


FIG. 205.

(14) The figure *PQRST* (Fig. 205) represents a king post (roof) truss supposed freely jointed at all the points and supported at *P* and *Q*. If

$PQ = 25$ ft. and $SPQ = 30^\circ$, find the stresses in all the bars when the loads at T, S, R, are 3, 4 and 3 tons respectively.

(15) Find the stresses in the bars of the roof truss shewn in Fig. 206 when loaded at R, S, and T with 3, 4 and 2 tons respectively.

$PT = TS = 8{\cdot}5.$
$PQ = 29{\cdot}5.$
$PU = 10{\cdot}8.$
$SU = 7{\cdot}8.$

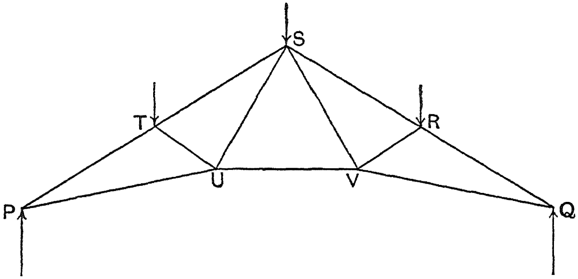


FIG. 206.

(16) Fig. 207 represents a queen post (roof) truss, supposed freely jointed at all points, supported at P and S, and loaded at T, U, V, W and X, with 2, 2, 3, 2 and 2 tons. Find the stresses in all the bars if

$$PX = XW = WV \text{ and } VPS = 30^\circ.$$

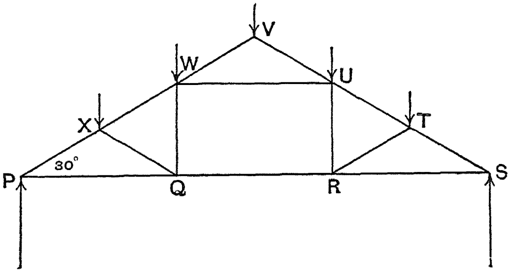


FIG. 207.

(The figure shewn is not rigid for anything but symmetrical loads; in practice the stiffness of the joints prevents distortion when the loading is not symmetrical.)

(17) *PQRSTU* (Fig. 208) represents a non-symmetrical king post truss loaded at *Q*, *S*, *T*, *U* with 1, 2, 2, 1 tons weight. Find, by the link polygon, the reactions at the points of support *P* and *R*, and then determine the stresses in the bars; given that $PR = 19$, $RT = 16{\cdot}6$, $PTR = 90^\circ$, and *QU* parallel to *RT*, *QS* parallel to *PT*.

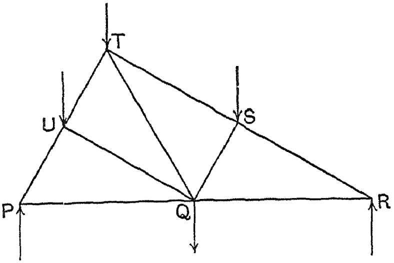


FIG. 208

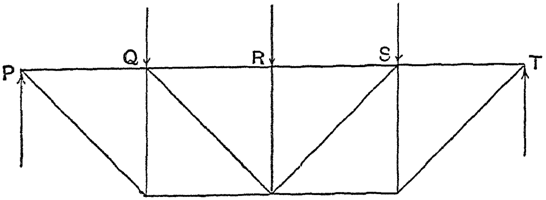


FIG. 209.

(18) Fig. 209 represents a short N girder consisting of right-angled isosceles triangles. Find the stresses in the bars when supported at *P* and *T*, and loaded at *Q*, *R*, *S* with 2, 3 and 4 tons weight.

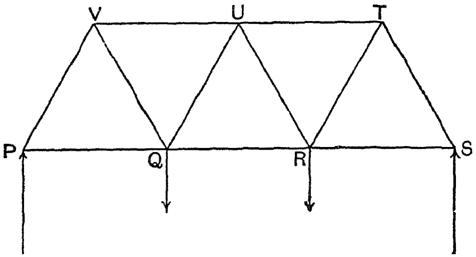


FIG. 210.

(19) *PQRSTUV* (Fig. 210) represents a short Warren girder of five equilateral triangles, loaded at *Q* and *R* with 7 and 10 tons. Find the reactions at the points of support *P* and *S*, and the stresses in all the bars.

(20) Find the stresses in the bars of the cantilever in Fig. 211, loaded at Q and R with 2 and 3 tons, and supported by a vertical wall at P and U; given

$PU=8$,
$PT=PS=8{\cdot}2$,
$UT=TS=3$,
$PQ=5$,
$QS=4{\cdot}5$,
$PR=8{\cdot}8$ ft.

(21) The queen post truss of Ex. 16, with a diagonal bar WR, is loaded at T, U, V, W and X with 2, 3, 3, 1·5 and 1·5 tons; find all the stresses.

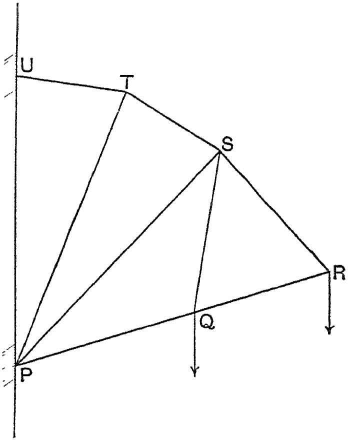


FIG. 211.

Weight of Bars in a Framework. In many engineering structures, the weight of the framework is very small compared with the loads it has to carry, and in such cases no appreciable error is made by neglecting the weights of the various parts. If the weights of the bars are not small in comparison with the loads, we shall suppose half the total weight of each bar concentrated at its ends.

Every particle of a body is acted on by a vertical downward force; the resultant of all these parallel forces is a parallel force through the M.C. of the body equal to the whole weight (see Chap. VIII., p. 307). This resultant is merely the single force which would produce the same motion as the actual forces on the particles, and it by no means follows that the other effects produced by it would be the same, *e.g.* as in bending the body or in producing internal stresses. Consider a vertical uniform column; the resultant force is a force through the mid-point of the column equal to the total weight. This force, if it acted at the M.C., would not produce any stress at all in the upper half, and a uniform stress for all sections in the lower half, whereas the actual stress must vary gradually from zero at the top to the total weight at the bottom. The average value of the compressive stress for the whole column would be the same in the two cases, viz. half the total weight.

The simplest way to suppose this average but constant stress produced would be to consider half the total weight concentrated at the top and half at the bottom, giving a constant stress measured by half the total weight and a pressure on the supporting ground equal to the total weight.

This is the approximation we shall adopt: by the stress in a bar of a framework, due to its own weight, we shall mean the average stress, produced by a load of half the total weight of the bar concentrated at each end.

When the bar is a sloping one, the weights of the various particles tend to bend the bar, and thus set up additional stresses. Except in Ch. IX. on Bending Moments, the bars of the frame will be supposed straight, and these additional stresses neglected.

A similar supposition will be made as to the effects produced by other forces acting on the bars at points other than the ends, viz. they will be supposed replaced by parallel and equivalent forces at the ends. Such suppositions have the additional advantage of making all the forces act at the joints of the framework, where they may be combined directly by the vector law of addition.

(22) Find the average stresses due to the weights of the bars set up in the Warren girder of Ex. 13, each bar weighing 530 lbs.

Reactions at Joints (two bars). Consider the simple cantilever *PQR* (Fig. 212), *PQ* and *PR* being uniform equally heavy rods of equal length. Suppose the weight of each bar to be *W* (15 lbs. weight). Replace the bars by weightless ones, having 7·5 lbs. weight concentrated at each end; then at *P* a vertically downward force of *W* (15 lbs. weight) acts. The average stresses *BC* and *CA* are obtained in the usual way, and are exactly the same as if the bars were weightless and a load **BA** were suspended from *P*.

To find the reactions of one body on another we may suppose the second body removed; then the force which has to be applied at the old point of contact, to maintain equilibrium, is the reaction of the second body on the first.

Hence the reaction of *cb* at *P* is found by supposing the bar *cb* removed, and seeing what force must be applied at *P* to keep the rest of the frame in equilibrium.

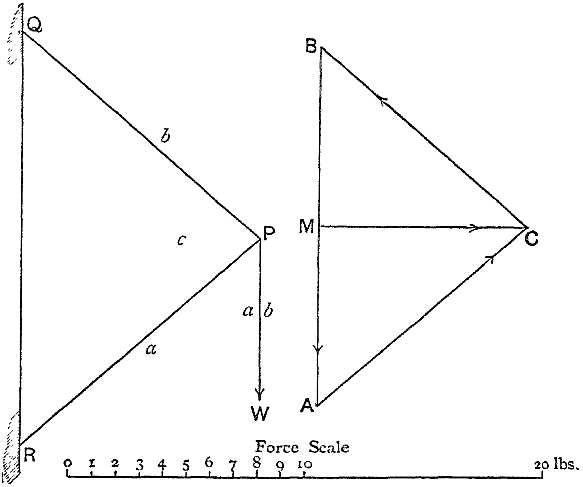


FIG. 212.

But if we remove *cb* we must suppose the load $\frac{1}{2}W$ also removed from *P*, hence we bisect *AB* at *M*, and **CB**+**BM** or **CM** is the reaction of *cb* on the pin at *P*; similarly, the reaction of the pin on *bc* is **MC**.

If the bars had been supposed weightless and a load *W* lb. suspended from *P*, then the stresses found in *bc* and *ca* would have been the same as before, but the reactions of the pin on *bc* and *ca* would have been quite different. For, on removing *bc*, no weight is taken away from *P*, and the reaction of *cb* on *P* would be simply **CB**. Similarly, on removing *ac*, the reaction on *ca* is seen to be **CA**.

(23) If $PQ=7\cdot2$ ft., $PR=5\cdot6$ ft. and $QR=6\cdot3$ ft., find the reaction at P on PQ if each bar weighs 37·24 lbs.

(24) As in the previous question if, in addition, a load of 40 lbs. hangs from P.

Reaction at a Joint (two unequal bars). PQR (Fig. 213) *is a small wall crane; the bars* PQ *and* QR *are uniform and the weights are 2 lbs. per foot. Find the pin reactions at* Q *given* PR = 9′, RQ = 3·9′ *and* QP = 7·1′.

Draw the total load vector **AB** of length proportional to half the weight of PQ and QR, and draw the stress diagram ABC as usual. Divide AB at D, so that $\frac{AD}{DB}=\frac{PQ}{QR}$. Join CD, and measure it by the force scale; it gives the reaction of the pin.

Suppose the bar PQ removed, then the force that must be applied at Q to maintain equilibrium is given by the resultant of the force **CA** and the weight **AD** (half the weight of PQ), *i.e.* by **CD**.

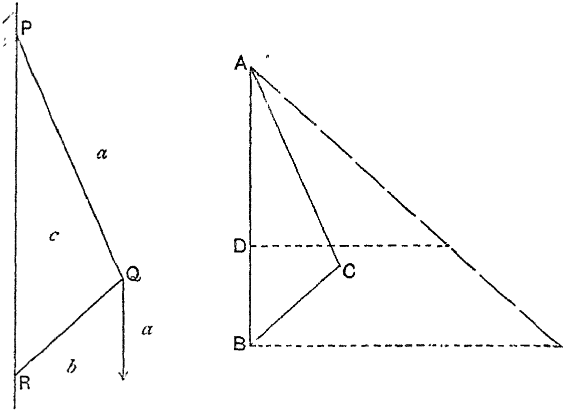


Fig. 213.

This vector **CD** gives, therefore, the reaction of the bar PQ on the pin at Q, and **DC** gives the action of the pin at Q on the bar PQ.

Similarly, if we suppose QR removed, we must remove the forces **BC** and **DB** and replace them by their resultant **DC**.

Hence **DC** is the reaction of the bar *RQ* on *Q*, and **CD** is the action of the pin at *Q* on the bar.

Evidently these results are consistent, for, the pin at *Q* being supposed weightless (or so small that its weight may be neglected), the total forces acting on it, viz. **DC** and **CD**, must be in equilibrium.

(25) Find the reaction of the pin on *bc* if a load equal to half the total weight of the bars be suspended from it.

Reactions at a Joint (three bars). If the bars have no weight, the reactions are simply along the bars; if they have, then the reactions of the pin on the three bars may be found as above by supposing the bars removed one by one, the reaction being the sum of the remaining forces, or the sum of the half weight of the bar and its force on the joint reversed in sense.

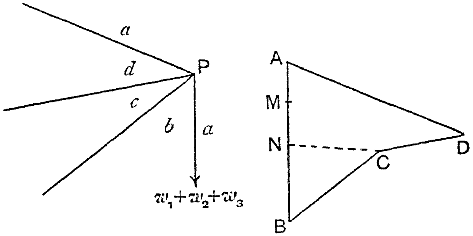


FIG. 214.

Thus, suppose *bc* (Fig. 214), *cd*, *da* are the bars, and w_1, w_2, w_3 their half weights, and that the vector polygon for *P* is as shewn, where $AM=w_3$, $MN=w_2$, $NB=w_1$.

To find the reaction of the pin at *P* on *bc*, suppose *bc* removed, *i.e.* remove **NB** and **BC** in the vector polygon; then **NC** is their resultant and **CN** the resultant of the remaining forces and is the reaction of the pin on *bc*.

Similarly, remove **DA** and **AM** (the force in *da* and half its weight); the sum of the remaining forces is **MD** and gives the reaction of the pin at *P* on *ad*.

Finally, remove **MN** and **CD** and the sum of the remaining forces is given by the sum of **DM** and **NC**.

Hence the sum of the three reactions of the pin is zero as it should be.

(26) Find the reactions of the pin at *V*, of the girder of Ex. 18, on the three bars meeting there, if the bars weigh 25 lbs. each.

EXAMPLE. AB *and* AC (Fig. 215) *represent heavy uniform beams, each* 7 *ft. long, pin-jointed at* A *and resting on rough walls at* B *and* C *in the same horizontal line. If* $BC = 9$ *ft. and the beams weigh* 500 *and* 700 *lbs., determine the reactions at* A, B, *and* C, *and the average stresses in* AB *and* AC.

It is assumed that the walls are sufficiently rough to prevent the beams sliding down.

Suppose the beams replaced by light rods loaded at their ends with 250 and 350 lbs. weight respectively. Then at *A* there is a load of 600 lbs. weight, at *B* 250 and at *C* 350 lbs. weight.

Draw the beam diagram to scale and letter the spaces as indicated; then draw the vector polygon *XYZU* for the loads. Then for the point *A*, *YVZ* is the stress diagram, and *VY* and *VZ* give the average values of the stresses in *AB* and *AC*. For *B*, *XYV* is the stress diagram. Since **XY** is the load at *B* and **YV** the push of the beam there on the wall, therefore **VX** gives the reaction of the wall at *B* on the beam *AB*. Similarly, **UV** gives the reaction at *C*. To find the reaction at *A* on *AB* suppose *AC* removed, then we must also suppose a weight of 350 lbs. removed from *A*; set off then $ZM = ZU$ and the resultant of the weight **MZ** and of the force **ZV** is the force **MV**, which is the reaction at *A* on the bar *AB*.

Measure the magnitudes of the reactions and the angles they make with the horizontal; measure also the average stresses in the beams.

Find the resultant of the weights of the beams (supposed to act through their mid-points) and shew by construction that the reactions at *B* and *C*, previously determined, are concurrent with it.

Of what general theorem is this concurrency an example?

If the walls were smooth and B and C were connected by a light rod, what would be the stress in that rod?

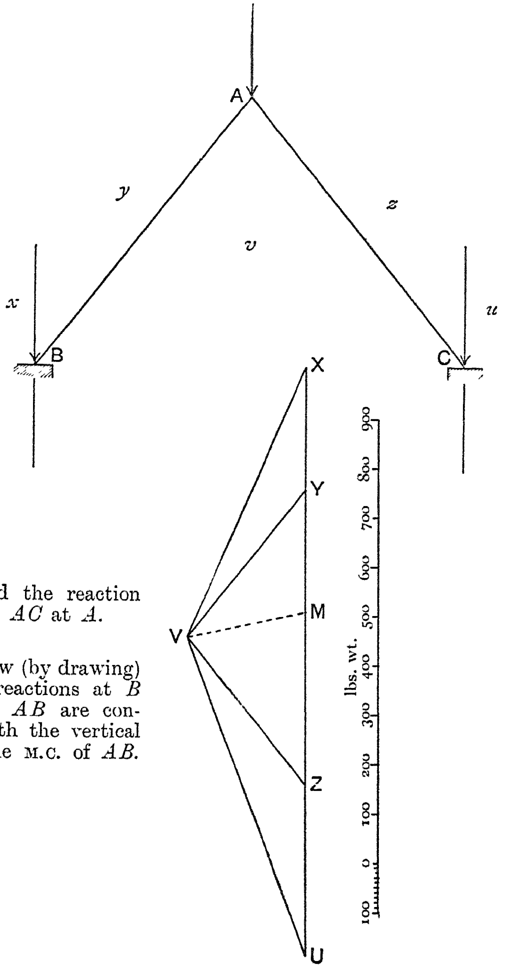

A
y
z
v
x
u
B
C
X
Y
M
V
Z
U
900
800
700
600
500
400
300
200
100
0
100
lbs. wt.

FIG. 215

(27) Find the reaction on the bar AC at A.

(28) Shew (by drawing) that the reactions at B and A on AB are concurrent with the vertical through the M.C. of AB.

Quadrilateral Frame and Reactions at Joints.

EXAMPLE. *Four equal bars, each of weight* 0·5 *lb. and length* 6″, *are pin-jointed together to form a square. It is suspended by one vertex and its form maintained by a light string connecting the upper and lower vertices. Determine the stresses in the string and rods and the reactions of the pins on the bars.*

Each bar may be supposed replaced by a weightless rod if half the weight be supposed concentrated at its end. This reduces the case to a frame loaded at the joints only. Further, if the string be supposed cut, the geometrical form will be maintained if we suppose a force equal to the tension in the string to be applied upwards at the lowest joint and downwards at the highest one. Letter the spaces as in the diagram (Fig. 216) and draw the vector polygon for the loads at Q, R, S, viz. 0·5 lb. at each; then the reaction, due to these, of the support at P is **AB** or 1·5 lbs. upwards.

At Q we have a force 0·5 lb. downwards (**BC**) and the forces due to the rods *bf* and *cf*. Draw, then, BF and CF (parallel to these rods) to intersect in F, the sense of the vectors being $BCFB$. Hence *cf* is in compression and *bf* in tension. Similarly, at S we get $DAED$ for the vector polygon.

At R we have then **FC**, **CD**, **DE** and the tension in *ef*. Join, therefore, E and F and $CDEFC$ is the vector polygon for the point. The tension in the string is twice the weight of a rod.

Tabulate the stresses and mark the bars which are in compression.

(29) Shew that the stress diagram for the point P gives results consistent with those already obtained. Find the stresses if $PR=2QS$.

For the reactions of the joints on the bars all we have to do is to suppose the corresponding bar removed and find the resultant of the forces acting at the joint, or find the force which would produce the same action on the joint as the bar, and then change its sense. It must be remembered that when a bar is supposed removed, we must take away the half load concentrated at the end under consideration.

At *Q*, suppose the bar *bf* removed, *i.e.* take away the pull **FB** and one half of **BC**; the resultant of the remaining forces **GC** and **CF** is **GF**, the reaction of the joint on the bar *bf*. The reaction on *cf* is similarly seen to be **FG**.

At *R*, suppose the bar *de* removed; then, in the stress diagram, take away **DE** and $\frac{1}{2}$**CD** and there remains *EFCH*, the resultant vector being **EH**. Similarly, the reaction on *cf* is found by removing **FC** and $\frac{1}{2}$**CD**, and is therefore **HF**

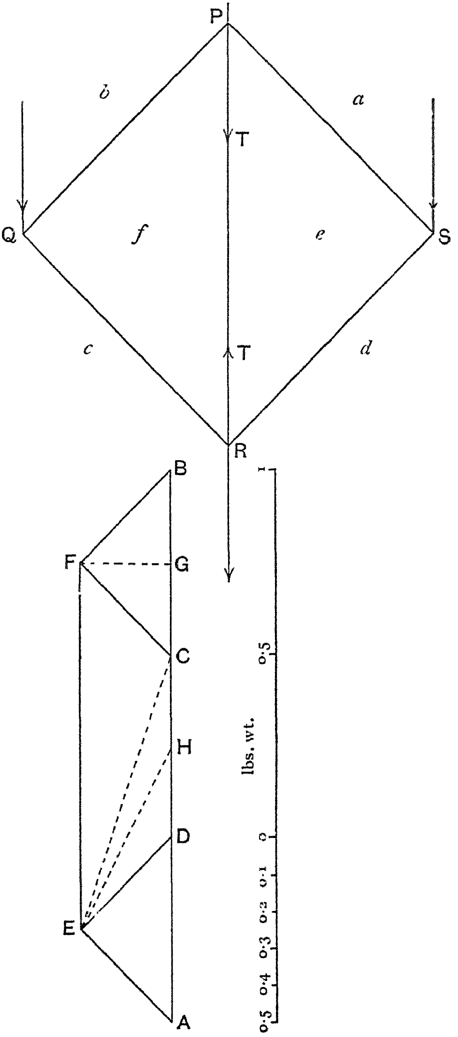


FIG. 216.

These results contrast with the case in which the joints are loaded and the weights of the bars are negligible, for in this latter case the action on any bar must be equal and opposite to the force of the bar on the joint, and is therefore given by the stress in the bar itself.

*Pentagonal Frame and the Reactions at the Joints.

EXAMPLE. *A regular pentagonal framework is suspended by one vertex, the regular form being maintained by a light string joining the top vertex to the middle point of the opposite side. Each bar weighs 1 lb.; find the stresses in the bars and string and the reactions of the pins on the bars.*

Proceed as in previous example, but now the pull due to the string at the mid-point of the lowest side must be replaced by equal upward forces at R and S (see Fig. 217) of magnitude to be determined. Letter the spaces as indicated, and draw the vector polygon $BCDEAB$ for the external forces. Then draw BCH for the point Q and EAF for T, then $HCDGH$ for R. The last gives **GH** as half the tension in the string. Complete the stress diagram and see that the part for the point P gives results consistent with those obtained before.

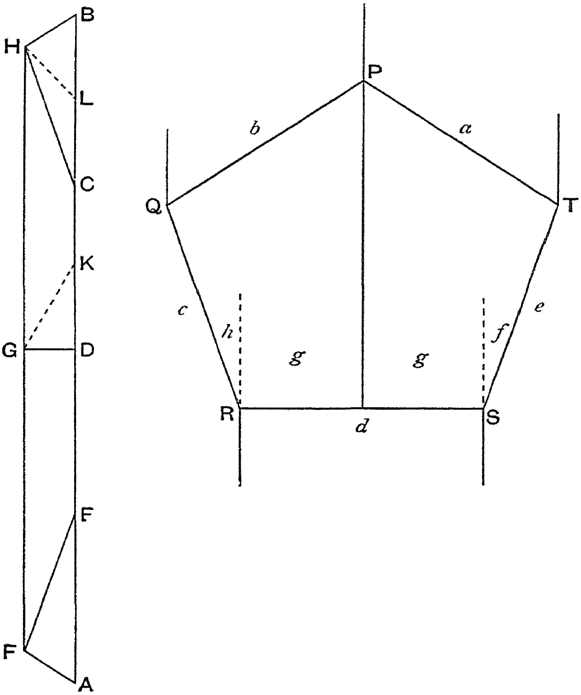
The reaction at Q on ch is **HL**, where L is the mid-point of BC. For the reaction at R on ch we must not only remove half the weight of the bar RS and its stress, but also half the tension of the string (since the string does not really pull at R); the reaction is therefore **KH** on ch, where K is the mid-point of CD.

Tabulate the stresses as usual.

(30) Two heavy uniform bars AB and BC are jointed together at B and to supports at A and C in the same horizontal line. If $AB=8$, $BC=4{\cdot}5$ and $AC=7$ ft., and the bars weigh 7 lbs. per ft., find the average stresses in AB and BC, and the reaction at B, A and C.

(31) A rectangular framework of four heavy bars is hung by one vertex, the rectangular state is maintained by a light non-vertical rod joining two vertices. The sides of the frame being 7 and 4·5 ft., and the bars weighing 25 and 16 lbs. respectively, find the average stresses in the bars and the compressive stress in the light rod.

(32) AB, BC and CD (Fig. 218) are three uniform heavy iron bars of weights 15, 10 and 15 lbs., hinged at A, B, C and D, and hung from A and D in the same horizontal line. Find the average stresses in the bars and the reactions at A, B, C and D if the bars weigh 1 lb. per ft.


B
H
L
C
K
G
D
F
F
A
P
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Q
T
c
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g
f
e
R
d
S

FIG. 217.

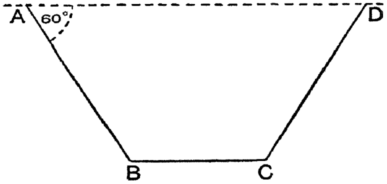


FIG. 218.

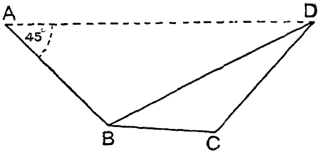


FIG. 219.

(33) $AB = 6$, $BC = 4{\cdot}5$, $CD = 6$, $BD = 9{\cdot}5$ and $DAB = 45°$. The shape in Fig. 219 is maintained by a light string BD; find the average stresses and the reactions if the bars weigh 3 lbs. per ft.

(34) Five equal bars are freely jointed to form a pentagon, which is suspended by one vertex. The frame is maintained in the regular pentagonal form by a light horizontal bar connecting two vertices. If the five bars weigh 7 lbs. each, find the stresses and the reactions at the joints.

(35) As in the previous example, only the bars are light and weights of 7 lbs. are suspended from the five vertices.

(36) A heavy triangular framework is suspended by one vertex, the sides are 3, 4 and 3·5 ft. long. The bars are uniform and weigh 0·2 lb. per foot. Find the position of equilibrium, and the stresses in the bars and the reactions at the joints.

(37) Three equal rods each weighing 2·5 lbs. are freely jointed together. The frame is supported at the mid-point of a horizontal side. Find the stresses in the bars and the reactions of the hinges on the bars.

(38) A regular hexagon of uniform bars, each of weight 3 lbs., is suspended from two vertices in the same horizontal line, the form is maintained by a light string connecting the mid-points of the top and bottom bars. Find the tension in the string and the reactions at the vertices.

(39) Find the average stresses in the bars of the king post truss of Ex. 13 due to the weight of the bars alone if they weigh 10 lbs. per ft.

The Funicular Polygon. The link polygon for like parallel forces can easily be constructed by means of weights and strings. The actual string polygon is called a **funicular polygon** (funicula = a little rope), and sometimes the meaning is extended to cover all the geometrical figures we have called link polygons.

EXAMPLE. BC *and* CD (Fig. 220) *are the vectors of forces whose axes are* bc *and* cd. *Choose any pole* O *between the perpendiculars at* B *and* D *to* BD. *Through the space* b *draw a line parallel to* OB, *through* c *a line parallel to* OC *and through* d *a line parallel to* OD.

In Fig. 220, PR_1R_2Q is the link polygon, P and Q being any points on the first and last lines drawn.

If the paper, on which the drawing has been made, be fixed to a vertical drawing board, so that bc is vertical, and a string of length $PR_1 + R_1R_2 + R_2Q$ be fixed to the board by stout pins at P and Q, then if weights proportional to BC and CD be fixed, hung by threads knotted to the string at R_1 and R_2 the forces acting on the string will cause it to be in equilibrium in the given position PR_1R_2Q.

To prove this requires merely a statement concerning the equilibrium of concurrent forces. R_1 is in equilibrium if three forces **BC**, **CO** and **OB** act there; the last two will therefore give the pulls which must be exerted on R_1 to balance **BC** and can be supplied by the tensions of the strings attached to R_1 in the positions given.

On joining PQ the link polygon is closed, and by drawing OA parallel to it in the vector polygon we get **AB**, the vertical reaction at P, and **DA** the vertical reaction at Q; moreover, **AO** gives the components of the reaction at P and Q along the line PQ. Hence, if PQ be a light rod suspended by vertical strings at P and Q, then AO would give the stress in PQ.

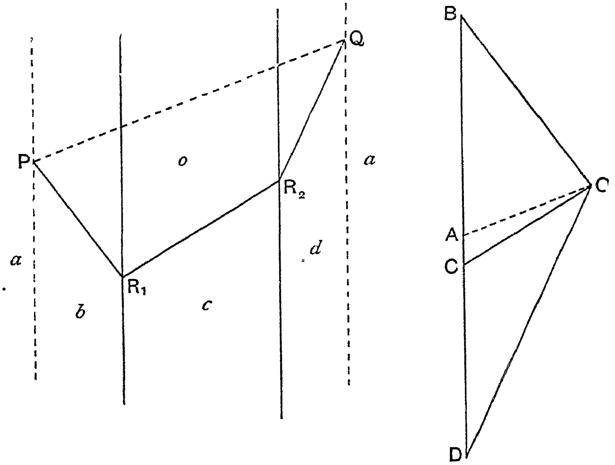


FIG. 220.

Notice carefully that we are able to draw the correct position of the funicular polygon corresponding to a certain arbitrary vector polygon, we can also, with certain exceptions, do the converse construction; *i.e.* given an arbitrary form PR_1R_2Q for the string, we can determine the weights BC and CD which will produce equilibrium and find the corresponding tensions in the string. It is essential that the form of the string polygon assumed be such that equilibrium can be obtained by tensile stresses only; hence in the arbitrary vector polygon the angles at B and D must be acute. Hence, the limitation as to the position of O. It is, of course, true that an infinite number of weights can be found, but the sets will all be proportional; *i.e.* taking the first, BC, arbitrarily, the other, CD, is fixed by the construction.

We cannot assume both the form of the string and the magnitudes of both the weights.

EXAMPLE. PQRS (Fig. 221) *represents a string attached to the points* P *and* S, P *being* 3 *ft. above* S, *and* PS = 15 *ft.*, PQ = 5, QR = 8·5 *and* RS = 7 *ft. At* Q *is tied a weight of* 4·7 *lbs.; find the weight which must be attached to* R *so that the angle* QRS *may be* 120°.

Draw first the polygon $PQRS$ to scale; to do this construct the triangle $Q_1R_1S_1$, where

$$Q_1R_1 = QR, \quad R_1S_1 = RS \text{ and } Q_1\hat{R}_1S_1 = 120°;$$

this gives the length of QS, and hence the point Q can be constructed.

Construct next the vector polygon for Q, viz. $BCAB$. At R we know the pull **AC** in ac, and can therefore complete the diagram for that point, viz. $ACDA$. The weight at R is therefore given by **CD**, the other lines give the tensions in the three parts of the string.

Notice that the horizontal components of all the tensions are equal.

(40) $PQRST$ (Fig. 222) is a funicular polygon, PT is horizontal; a load of 5 lbs. is suspended at Q; find the loads at R and S necessary to maintain the given shape. $PQ = 3{\cdot}5''$, $QR = 4{\cdot}5''$, $RS = 6''$, $PT = 17''$.

(41) If the strings PQ and ST be suspended from a light rod PT hung by vertical strings, find the tension in the strings and the compression in the rod PT.

(42) The tension in PQ being 18 lbs., what must be the weights suspended from Q, R and S?

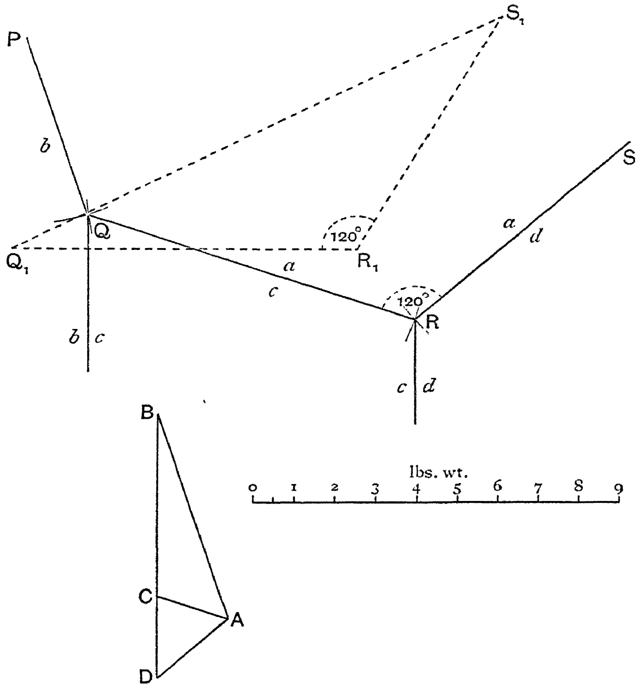

S₁
P
b
S
a
d
Q
120°
Q₁
R₁
a
c
120°
R
b
c
c
d
B
lbs. wt.
0 1 2 3 4 5 6 7 8 9
C
A
D

FIG. 221.

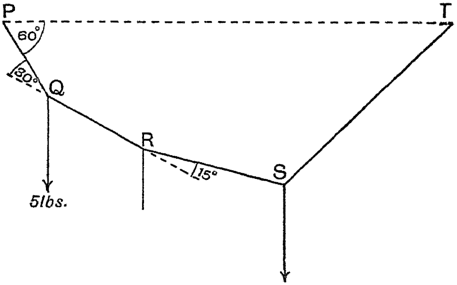

P
T
60°
30°
Q
R
15°
S
5lbs.

FIG. 222.

Funicular Polygon for Equal Loads, the axes being at equal distances.

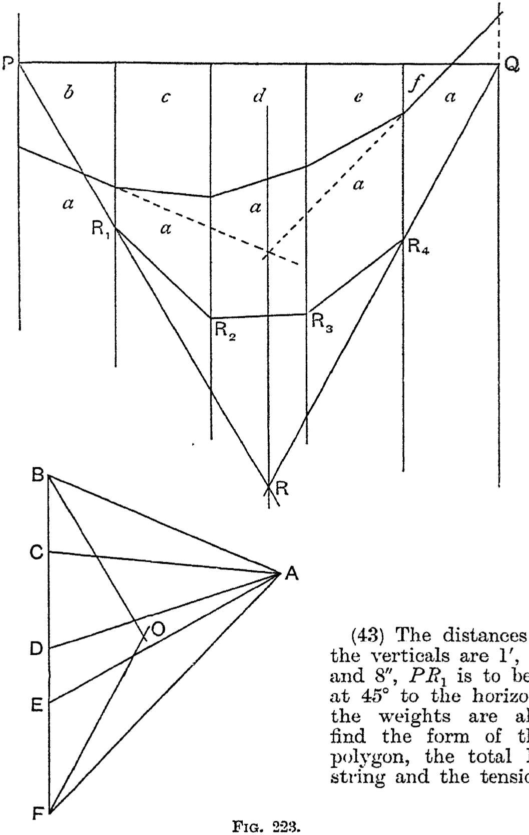
EXAMPLE. P *and* Q (Fig. 223) *are two fixed points in a horizontal line, distant apart* 2·5 *ft.; a string is to be fixed to* P *and* Q, *and loaded with weights* 2, 2·5, 1·5, 3 *lbs., whose axes are equidistant (viz.* 6″*). The part of the string attached to* P *is to make* 30° *with the vertical. Find the length of string necessary, the form of the funicular polygon and the points on it where the weights have to be fastened.*

Draw first the vector polygon of the weights *BCDEF*, choose a pole *A* for the vector polygon, and find the position of the resultant by the link polygon. Find the point *R* of intersection of the known direction of the string at *P* and the axis of the resultant. Join *R* to *Q*; then *RQ* is the direction of the string at *Q*. Draw, then, *BO* and *FO* parallel to *PR* and *QR* to intersect at *O*. *O* is then the pole of the vector polygon, and by joining *O* to *CDE* we get the directions of the remaining segments of the string. Draw these directions on the link polygon, and measure the segments between the given vertical lines.

It may appear at first sight that, although we can determine the point *O* in the manner given, yet when we come to construct the corresponding link polygon —PR_1 parallel to *OB*, R_1R_2 parallel to *OC*, R_2R_3 parallel to *DO*, R_3R_4 parallel to *EO*—the last link through R_4 parallel to *OF* would not necessarily go through *Q*.

A little consideration of the properties of the link and vector polygons will shew the necessity of this.

We have a number of parallel forces in *bc*, *cd*, *de* and *ef*, and **BF** is their vector sum; through *b* draw PR_1 parallel to *BO*, and suppose in PR_1 two forces **OB** and **BO**, differing only in sense, to act. The resultant of **OB** and **BC** is **OC** along R_1R_2, and so on to the resultant of **OB**, **BC**, **CD**, **DE** and **EF** is **OF** acting along R_4Q_1, where Q_1 is the point on the vertical through *Q*, where R_4Q_1, parallel to *OF*, cuts it. Combine, then, **BO** in PR_1 with **OF** to a resultant **BF** acting through the point of intersection of PR_1 and Q_1R_4. But the point in PR_1, where this resultant cuts it, has been determined already, viz. the point R, and RQ is parallel to OF, hence RQ and RQ_1 must be both parallel to OF, *i.e.* Q_1 is at Q.



(43) The distances apart of the verticals are 1′, 6″, 4″, 4″ and 8″, PR_1 is to be inclined at 45° to the horizontal, and the weights are all equal; find the form of the string polygon, the total length of string and the tensions.

FIG. 223.

(44) The loads being 2·1, 2·2, 1·8 and 2·9 lbs., and the vertical axes equidistant (6″), find the form of the link polygon and the total length of string necessary if *ad* is to be horizontal; PR_1 makes an angle of 45° with vertical and Q is fixed only inasmuch as it must lie in a certain vertical.

(45) As in Ex. 43, but now *ac* and *ae* are to be equally inclined at 45° to the horizontal.

The theory of the funicular polygon is of great practical importance, since it is immediately applicable to the determination of the form and stresses of suspension bridges.

Funicular Polygon satisfying prescribed conditions. A funicular polygon is to be constructed between two points, P and Q, distant 3 ft. apart in a horizontal line. It is to be loaded with seven equal weights of 5 lbs., the axes are to be equidistant and the lowest vertex, V, $11\frac{1}{4}$ inches below M, the mid-point of PQ.

Since the axes and loads are symmetrical with regard to MV, only half the funicular and stress diagrams need be drawn.

Set off PM and MV (Fig. 224) to scale (say half full size) and letter the spaces.

Set off to scale AB, BC, CD for the loads in ab, bc, cd, and $DK=\frac{1}{2}AB$; draw KO perpendicular to AK; then the pole of the vector polygon must be in KO. This follows from the known symmetry of the funicular polygon. Take any point O in KO as pole, and proceed to draw the link polygon, starting at V, viz. VV_1 parallel to OD, V_1V_2 parallel to OC, V_2V_3 parallel to OB and V_3V_4 parallel to OA.

If V_4 is at P we have solved the problem.

Evidently by taking different poles along KO, and constructing the corresponding link polygons, we should get a series of points like V_4, and by trial we could find the position of the pole so that V_4 should be at P.

Through V draw a line Vu_1 parallel to PM and produce the links V_2V_1, V_3V_2, V_4V_3, to cut it at u_1, u_2, u_3.

Join P to u_3 cutting ab in W_3; join W_3 to u_2 cutting bc in W_2; join W_2 to u_1 cutting cd in W_1; and, finally, join W_1 to V. Then $PW_3W_2W_1V$ is the half funicular polygon required.

Through A, B, C and D draw lines parallel to PW_3, W_3W_2, W_2W_1, W_1V; these should all be concurrent to a point O_1 in KO.

Measure AO_1, BO_1, CO_1 and DO_1 on the force scale; these are the tensions in the various parts of the string polygon.

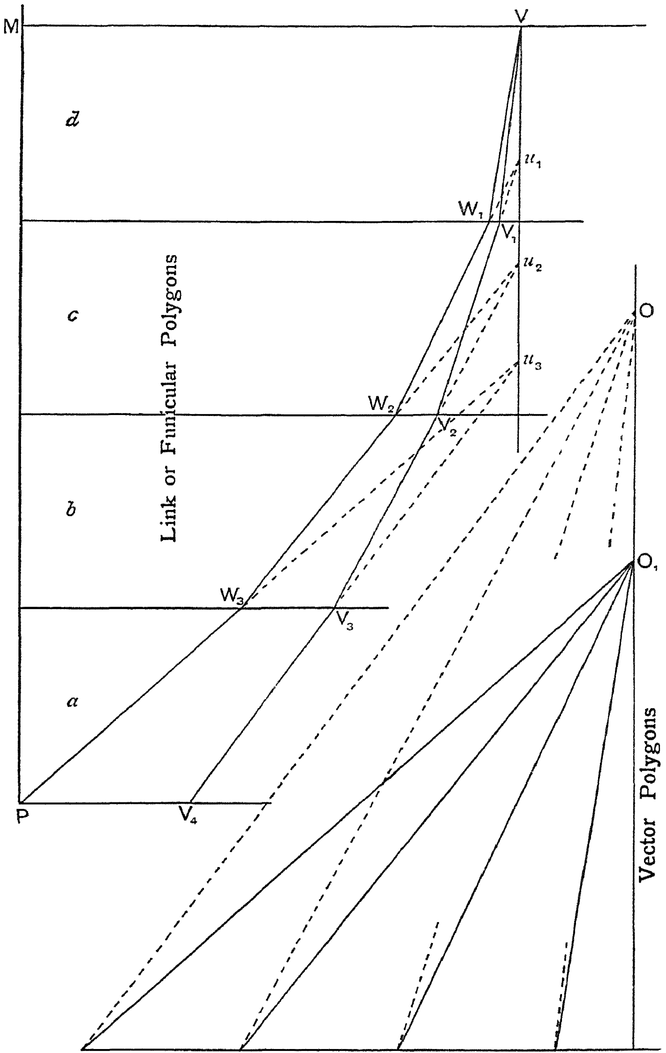

M
V
d
u_1
W_1
V_1
u_2
Link or Funicular Polygons
c
O
u_3
W_2
V_2
b
O_1
W_3
V_3
a
P
V_4
Vector Polygons
A
B
C
D
K

Fig. 224.

***Proof.** Suppose that corresponding to the two poles O and O_1 there are two link polygons $VW_1\dots$ and $VV_1\dots$. Through V draw a line Vu_1 parallel to OO_1, cutting $W_2W_1, \dots$ in u_1, u_2 and u_3.

Let **OO$_1$** be the vector of a force in Vu_1, then **OO$_1$**, **O$_1$D** and **DO** in Vu_1, VW_1 and VV_1 are in equilibrium. To **O$_1$D** add **DC** and to **DO** add **CD**, both forces acting in cd, then **OO$_1$**, **O$_1$C** and **CO** are in Vu_1, W_2u_1 and V_2V_1 are in equilibrium. But three forces in equilibrium must be concurrent, hence V_2V_1 must pass through u_1. A similar argument shews that V_3V_2 and V_4V_3 pass through u_2 and u_3 respectively.

Generally, then, we see that wherever the pole O of the vector polygon be taken in the line KO, the corresponding lines of the link polygon must intersect on the line through V parallel to OO_1. This is really a special case of a more general theorem:

Given the axes and vectors of a set of forces, if two link polygons be drawn for poles O *and* O$_1$, *then the corresponding sides of the link polygons intersect on a line parallel to* OO$_1$.

The Funicular Polygon and Parabola. *The vertices* V, W$_1$, W$_2$, W$_3$, ..., *of p.* 243, *lie on a parabola having* V *as its vertex and* VM *as its axis of symmetry. Construct the links of the funicular as follows:*

Fig. 225 shews the right-hand half of the funicular of which the left half has already been constructed. P_1M_1 or ef, P_2M_2 or fg are the load lines on which the vertices are to be found.

Join VQ and construct the vertices on P_1M_1, P_2M_2, ... of the parabola going through Q (see pp. 29-31).

Curves whose equations are like $y=x^2$, $y=2x^2$ or $y=Ax^2$ are called 'parabolas.'

Go vertically from M_1 to Q_1, horizontally to R_1 and mark P_1 where VR_1 cuts M_1P_1. Similarly, determine the points P_2 and P_3.

Set off the load vectors vertically downwards DE, EF, FG and GH, and mark K, the mid-point of DE. Then draw, through E, F, G and H, lines parallel to the links of the funicular and see that they all intersect (or nearly so) on the line KO perpendicular to DH.

(46) Draw to scale the funicular polygon joining two points P and Q in the same horizontal line, $PQ=2\cdot5$ ft. The string is to be loaded with six equal weights (1·5 lbs. each), and the axes are to be equidistant and the lowest link is to be 2 ft. below PQ. Find the tensions in the various string segments.

(In this case the lowest link is horizontal, and if the pole of the vector polygon be chosen on the perpendicular through the mid-point of the resultant load vector, the corresponding sides of the two link polygons must intersect on the lowest link.)

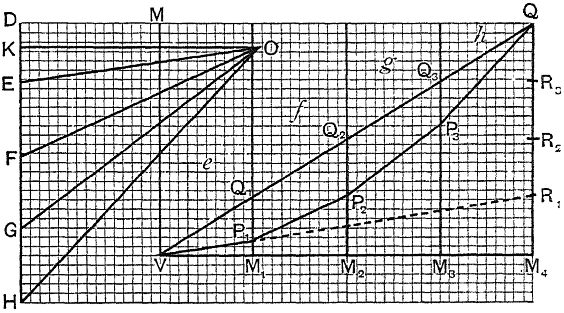


FIG. 225.

*(47) Draw to scale the funicular polygon joining two points P and Q where $PQ=4$ ft., and makes an angle of 30° with the horizontal through P and below it. There are to be six equal loads in equidistant axes, and the first link is to make an angle of 40° with the vertical. Find also the tensions.

(In this case, through the beginning of the total load vector draw a line making 40° with the vertical and take the pole on this. Draw the first link of the funicular through P, then wherever the pole of the vector polygon be taken on the line drawn, the corresponding sides of all funiculars starting at P must intersect on the first link. Hence it is easy to draw that particular funicular which will pass through Q.)

*(48) Funicular as in Exercise 47, but the vertex bearing the middle load is to be 2 ft. below the mid-point of PQ.

***The Funicular Polygon and Parabola. The vertices of a funicular polygon for which the loads are all equal and at equal distances apart lie on a parabola.**

(a) Suppose the number of loads odd.

Let w be the load at each vertex, h the distance apart of the loads, V the lowest vertex and Q any other, say the n^{th}. Take the axes of coordinates as the horizontal and vertical through V

(Fig. 226) and suppose the coordinates of Q, x and y. Between Q and V are $\overline{n-1}$ loads and n spaces, hence the resultant load is $\overline{n-1}w$ whose axis is mid-way between V and Q.

The part of the chain between V and Q is in equilibrium under the load $\overline{n-1}w$ and the tensions in the lowest and highest links. Hence, since three forces in equilibrium must pass through a point, these links must intersect on the axis of the resultant load, viz. at M in Fig. 226 whose abscissa is $\frac{x}{2}$.

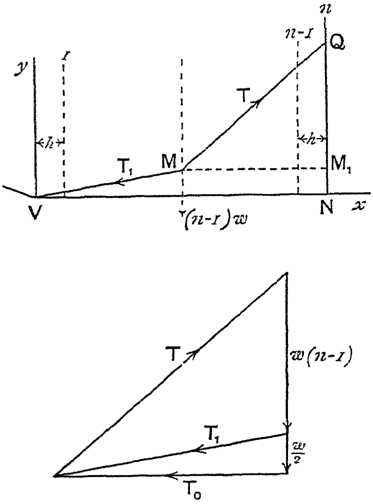


FIG. 226.

In the stress diagram T and T_1 are the supposed tensions in the links, then T, T_1 and $w(n-1)$ form the vector triangle for M.

For the point V, T_1, w and by symmetry a force of magnitude T_1 form the vector triangle. Hence T_0 denoting the horizontal component of all the tensions, we have T_1 equivalent to T_0 and $\frac{1}{2}w$.

Comparing the similar triangles of the funicular and vector polygons we have at once

$$\frac{y - M_1N}{\frac{x}{2}} = \frac{w(n-\frac{1}{2})}{T_0},$$

$$\frac{M_1N}{\frac{x}{2}} = \frac{\frac{w}{2}}{T_0}.$$

Therefore, by the properties of fractions (adding numerators and denominators),

$$\frac{y}{x} = \frac{wn}{2T_0}.$$

But $$nh = x;$$

$$\therefore\ y = \frac{w}{2hT_0}x^2.$$

This is the equation to a parabola, the axis of y being the axis of symmetry. Hence all the vertices lie on the parabola, for the equation remains exactly the same whatever particular value n may be supposed to have.

When the dip and span of the funicular and the number of loads are known, T_0 can be calculated.

Let $2s =$ the span,

$d =$ the dip,

$N =$ number of loads $\left(h = \frac{2s}{N+1}\right)$.

Then $$d = \frac{w}{2hT_0}s^2 = \frac{w(N+1)s}{4T_0}$$

or $$T_0 = \frac{w(N+1)}{4} \cdot \frac{s}{d}.$$

(*b*) Suppose the number of loads even, then the middle link is horizontal. First take the origin at the middle point of this link. Then for the equilibrium of the piece between V and Q the link through Q must, when produced, pass through M the mid-point of the space between the 1st and $\overline{n-1}$th verticals.

In Fig. 227,

$$x=(n-1)h+\frac{h}{2},$$

and $$MN=\frac{nh}{2};$$

$$\therefore \frac{y}{\frac{nh}{2}}=\frac{w(n-1)}{T_0};$$

$$\therefore y=\frac{wnh}{2T_0}(n-1).$$

But

$$(n-1)h=x-\frac{h}{2},$$

and $$nh=x+\frac{h}{2};$$

$$\therefore y=\frac{w}{h2T_0}\left(x^2-\frac{h^2}{4}\right);$$

$$\therefore y+\frac{wh}{8T_0}=\frac{w}{2hT_0}x^2.$$

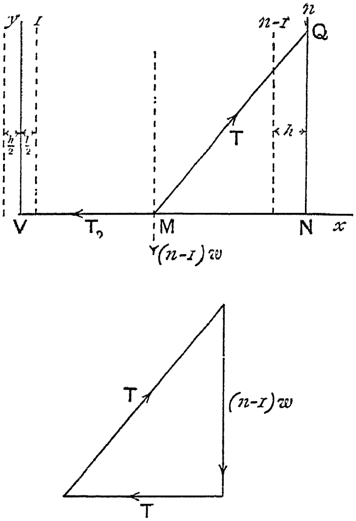


FIG. 227.

If the origin O be taken at a distance $\frac{wh}{8T_0}$ below V, the equation becomes $y=\frac{w}{2hT_0}x^2$, a parabola.

***Method of Sections.** If any closed curve be drawn cutting some of the bars of a frame in equilibrium, then the part inside the curve may be looked upon as a rigid body in equilibrium, under the action of the known external forces on that part of the frame, and certain forces acting along the bars at the points where the curve cuts them. If the forces necessary to produce equilibrium acting along these bars can be found, they will give the magnitude of the stresses in those bars, and whether the stresses are tensile or compressive can be determined by the senses of the forces.

Fig. 228 represents an ordinary queen post truss. If we consider the part enclosed by the dotted curve on the right, it will be in equilibrium under the action of the external forces in *oa*, *ab* and *bc*, and certain forces in *ch*, *hi* and *io*. Suppose these bars cut; then, to maintain this right-hand portion in equilibrium, we must apply to the cut surfaces of the bars forces equal in magnitude to the stresses in them before cutting, and of the proper sense. For instance, if *ch* be in compression we must take the sense of the force at the cut surface from left to right; if in tension, from right to left. Conversely, if we can find the forces in these cut bars which will keep the right-hand portion in equilibrium, these forces with their proper senses will give the stresses in the bars.

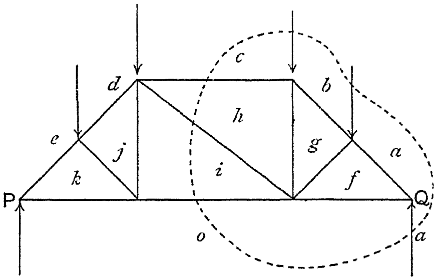


FIG. 228.

If the curve can be drawn so as to cut only three non-concurrent and non-parallel forces, we can always find by this means the stresses in the three bars. (In special cases we may be able to find the stresses in one or more bars when more than three are supposed cut).

The method of sections enables us (i) to test the accuracy of the stresses already found by independent means; and (ii) to draw stress diagrams for various frames in which the ordinary method of procedure fails.

To determine the stresses, we either use the construction for resolving a known force into three components acting in three non-concurrent and non-parallel lines, or the construction for finding the sum of the moments of a number of forces. The latter method will be explained in Chap. X.

*EXAMPLE. *Determine the stresses in the bars* ch, hi *and* io *of the queen post truss* (*Fig.* 229) *if* VZ = 40, VW = 12, UW = 12 *ft.*, *the loads in* ed, dc, cb *and* ba *being* 0·5, 0·6, 1·5 *and* 1 *tons.*

Draw the truss to scale and the load and reaction lines.

Draw the vectors **ED, DC, CB** and **BA** of the loads in *ed*, *dc*, *cb* and *ba*. Take any pole *P* of the vector polygon and construct the link polygon in the usual manner. Close the link polygon and draw *PO* in the vector polygon parallel to the closing line.

Draw any closed curve cutting the bars *ch*, *hi* and *io*. Then the forces acting on the enclosed body are **OE, ED** and **DC** and the forces in the bars supposed cut. For these forces the first line of the link polygon drawn is parallel to *PO*, and the last is parallel to *PC*; these lines intersect (when produced) at *R*, consequently the resultant of **OE, ED** and **DC**, viz. **OC**, acts through *R* parallel to *OC*.

Produce *io* to cut the axis of this resultant in *Q*. Join *Q* to *U*, the point of intersection of *ih* and *ch*. From *C* and *O* draw lines parallel to *io* and *QU*, viz. *CX* and *OX*. Then **CX** and **XO** are in equilibrium with **OC**, their axes being *io*, *QU* and *RQ*.

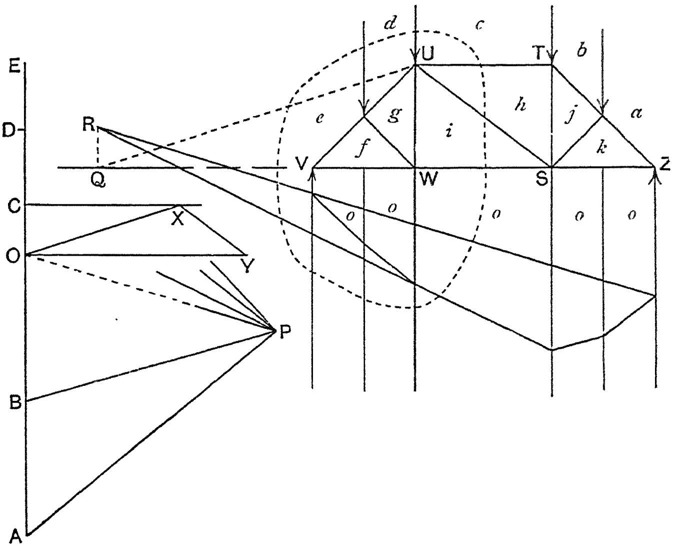
From *O* and *X* draw *OY* parallel to *ch* and *XY* parallel to *ih*; then **OC** acting along *RQ* is in equilibrium with **CX, XY** and **YO** acting along *io*, *ih* and *ch* respectively.

Since **CX** pulls at the cut end of the bar *io* (enclosed by the curve in Fig. 229) *io* must be in tension. Similarly *ih* is in tension and *ch* is in compression.

When the angle is small between the sides of the link polygon, for which the point of intersection is required, the method becomes untrustworthy. In that case the moment method explained in Chap. X. is used.

Cases in which the method of sections enables us to draw stress diagrams for which the ordinary method fails will be considered also in Chap. X.

(49) Draw the frame in Fig. 230 and determine the stresses in *bg*, *gf* and *fd* by the method of sections. The load at $P=1$ ton and at $Q=0·5$ ton. $RS=30$, $RP=21$, $RQ=QP=11$ ft.


E
D
C
O
B
A
R
Q
X
Y
P
d
c
U
T
b
e
g
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h
j
a
f
k
V
W
S
Z
o
o
o
o
o

FIG. 229.

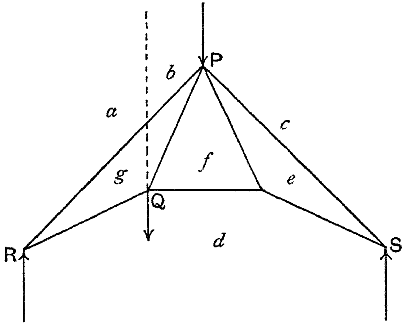
P
b
a
c
f
g
e
Q
d
R
S

FIG. 230.

(50) Find the stresses in *bj*, *jk* and *ko* in the queen post truss of Fig. 229.

MISCELLANEOUS EXAMPLES. VI.

1. *ABC* is a triangle having a right angle at *B*. The sides $AB = 12''$, $BC = 5''$. A force of 52 lbs. weight acts from *A* to *C*, one of 24 acts from *B* to *A* and one of 27 from *C* to *B*. Find the magnitude and line of action of the resultant. Draw the figure to scale and exhibit the resultant. (Inter. Sci., 1904.)

2. *ABC* is a light frame in the form of a right-angled triangle, *A* being the right angle. It can turn in a vertical plane about a fixed horizontal axis at *A*; and, when a weight *W* is suspended from *C*, the corner *B* (which is vertically below *A*) presses against a fixed vertical plate. Find graphically the stress in each rod and the reactions at *A* and *B*. (Inter. Sci., 1906.)

3. The pair of rafters shewn in Fig. 231 carry a load equivalent to 150 lbs. at the middle of each. Find the direction and magnitude of the thrust on each wall.

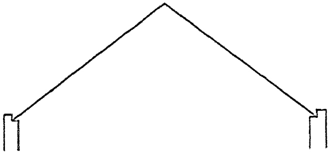

FIG. 231.

If the walls are relieved of horizontal thrust by a rod joining the lower ends of the rafters, what is the pull on the rod? (War Office, 1904.)

4. A triangular frame *ABC* can turn freely in a vertical plane about *A* (the right angle). The side *AB* is horizontal, and the corner *C* rests against a smooth vertical stop below *A*. Find, graphically or otherwise, the stresses in the various bars due to a weight *W* suspended from *B*. $AB = 3$ ft., $AC = 1$ ft., $W = 50$ lbs. weight. (Inter. Sci., 1903.)

5. The beams *AB* and *AC* (Fig. 232) are part of a roof and carry a load of 160 kilogrms. at the ridge. The span of the roof is 10 metres, and the beams make 28° with the horizontal. Find the thrust this load causes on each wall.

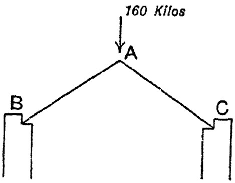

FIG. 232.

If the walls are relieved of the outward thrust by a rod joining the ends *B* and *C* of the beams, what is the thrust in this rod, and what thrust do the walls still bear? (Military Entrance, 1906.)

6. A heavy straight rod *AB*, 12 ft. long, turns on a pivot at *A*, and is supported in a horizontal position by a vertical force of 15 lbs. weight applied at *B*. If the weight of the rod is known to be 90 lbs., find the pressure on the pivot and the position of the centre of gravity.
(Naval Cadets, 1904.)

7. A heavy uniform beam *AB* rests in a vertical plane against a smooth horizontal plane *CA* and a smooth vertical wall *CB*, the lower extremity *A* being attached to a cord which passes over a smooth pulley at *C* and sustains a given weight *P*. Find the position of equilibrium.
(B. of E., II.)

8. Find the stresses in the bars of the short Warren girder *PQRST* figured (Ex. 12, p. 222). Loads of 2 and 1 tons act at *P* and *T*, and the frame is freely supported at *Q* and *S* in same horizontal line.
(Inter. B.Sc. (Eng.), 1905.)

9. *ABCD* is a thread suspended from points *A* and *D*, and carrying a weight of 10 lbs. at *B* and a weight *W* at *C*; the inclination to the vertical of *AB* and *CD* are 45° and 30° respectively, and *ABC* is an angle of 165°. Find, by construction or calculation, *W* and the tension of *BC*.
(B. of E., II.)

10. A uniform bar is bent into the shape of a V with equal arms, and hangs freely from one end. Prove that a plumb line suspended from this end will cut the lower arm at $\frac{1}{3}$ of its length from the angle. (B.Sc., 1904.)

11. Fig. 233 shews (drawn to scale) a rectangular framework of four bars freely hinged. It is supposed to lie on a smooth table with *AB* fixed. A piece of elastic cord is stretched, and its ends fastened at *E* and *F*. For what positions of *E* and *F* will the frame remain rectangular, for what positions will the frame move to the right and for what positions to the left? Justify your statements.

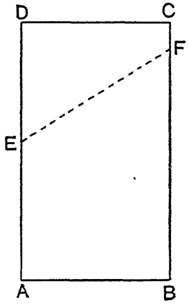

FIG. 233.

If the frame is held rectangular by a pin driven through the corner *C*, and the elastic cord stretched till it exerts a pull of 10 lbs., and fastened to the frame at the corners *A* and *C*, what force does the pin at *C* exert on the frame? (Military Entrance, 1905.)

12. Fig. 234 is a rough sketch of a crane. If the weight hanging at the point *A* is 1000 lbs., find, graphically or otherwise, the forces acting along the bars *AC* and *AB*; and if the post *BC* is free to move in the vertical plane, find the pull in the tie *CD* which will prevent the crane from toppling over.
(Military Entrance, 1905.)

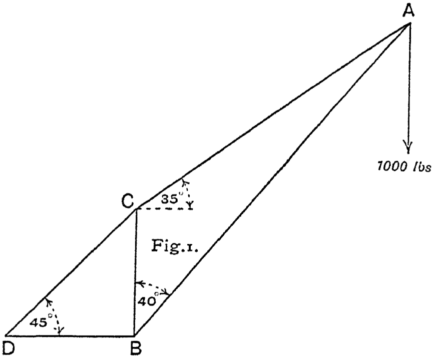

FIG. 234.

13. *AB* and *BC* are two uniform rods fastened by smooth joints to each other at *B*, and to fixed points *A* and *C*; the point *C* being vertically above *A* and $CA = AB$; given that the weight of *BC* is twice that of *AB*, find the reaction at *C* and *A* by which the rods are supported.
(B. of E., III., 1904.)

14. Draw a circle and from a point *A* outside the circle draw two tangents and produce them to *B* and *C*. Suppose that *AB* and *BC* are two equal uniform rods connected by a smooth hinge at *A* at rest on a smooth vertical circle; find the position of the rods when *AB* is 10 times as long as the diameter of the circle. (B. of E., III.)

15. *A*, *B*, *C* are fixed smooth points, such that *ABC* is an equilateral triangle, with *BC* horizontal and above *A*. A fine thread is fastened to *A*, passes over *B* and *C* and carries a weight of 10 lbs.; find the pressures on *B* and *C* produced by the weight. (B. of E., I., 1903.)

16. A uniform rod *AB* can turn freely, in a vertical plane, about a hinge at *A*; the end *B* is supported by a thread *BC* fastened to a fixed point *C*; *AC* is horizontal and equal to *AB*. Draw a triangle for the forces which keep the rod at rest, and shew that in any inclined position of *AB* the reaction of the hinge is greater than the tension of the thread.
(B. of E., II., 1903.)

17. *A*, *B*, *C* and *D* are four points in a horizontal line. Two weights, *P* and *Q*, are to be supported by three strings, *AP*, *PQ* and *QD*, in such a way that *P* is vertically below *B* and *Q* below *C*. Give a construction for finding suitable lengths of string for the purpose.

Also shew that in any configuration in which the stated condition is satisfied, the string *PQ* will intersect *AD* produced in a fixed point.
(Patent Office, 1905.)

18. The outline of a crane is as follows: *ABC* is a right-angled triangle, *AB* being horizontal and *AC* vertical, and *AB* is equal to *AC*; on the other side of *AC* is a triangle *ADC*, the angle *A* being 45° and the angle *C* being 120°. A load of 10 tons hangs from *D*, and the crane is supported at A and anchored at B. Find the reactions at the supports, and the stresses in AC, CD and BC. (Patent Office, 1905.)

19. Draw five parallel lines at distances apart 4, 5, 3 and 6 inches from left to right. Construct a funicular polygon for loads $2W$, W, $3W$, $2W$ and W suspended from vertices on these lines, the sides of the polygon which stretch across from the middle line to its neighbours being each inclined at 60° to the vertical. (Inter. Sci. B.Sc. (Eng.), 1904.)

20. Draw the stress diagram for the following truss, Fig. 235, indicating which members are in tension and which in compression. Assume equal vertical loads. (Admiralty Exam., Assistant Engineer, 1904.)

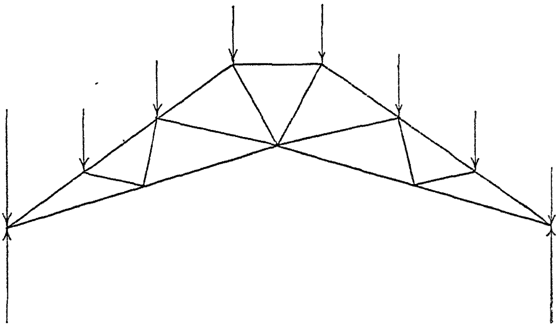

FIG. 235.

21. A king post roof truss is loaded with 1 ton at each of the five joints; find graphically the amount and nature of the force acting in each member. Span $=30$ ft., pitch $=7' \, 6''$.

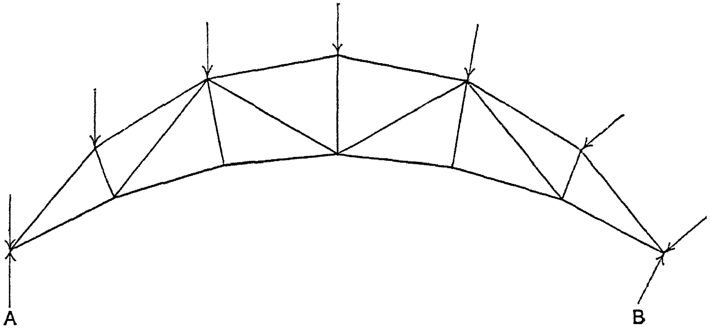

FIG. 236.

22. Draw the stress diagram for the frame shown in Fig. 236, assuming the joints to be loaded with equal weights in the direction of the arrows. The arrows A and B indicate the direction of the reactions.

(Admiralty, 1905.)

CHAPTER VII.

FRICTION.

HOWEVER carefully the Experiments I.-VI. are performed, there are sure to be some slight divergencies from the results stated. These discrepancies are chiefly due to the friction at the pulley, and but to a very small extent to the weight of the ring or card.

Resistance to Motion. In all cases where one body slides or tends to slide on another, a resisting force called **friction** is called into play. This force has the same direction as that of the motion, or attempted motion, but is of the opposite sense.

EXPT. XII. *Apparatus*: A board with a smoothly running pulley screwed or clamped on to one end; an open box with a hook for fastening a string a set of weights and a scale pan. A board of varying width is best, as this will give different areas of contact between the box and the board.

Arrange the board horizontally and the box, etc., as in Fig. 237. See that the hook is screwed in a position so that the pull of the string is horizontal. Put a 500 gramme weight in the box and load the scale pan 20 grammes at a time until motion ensues. Find the greatest weight that can be put into the pan without moving the box.

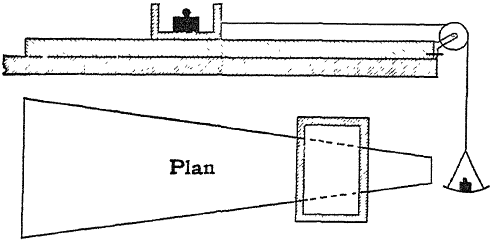

FIG. 237.

Starting with no weights in the box, increase the load by 100 grammes at a time up to 800, and find in each case the maximum weight that can be placed in the scale pan without moving the box.

Tabulation of Results. Add to each load put in the box the weight of the box itself, and similarly the weight of the scale pan to its load in each case, and tabulate the results as indicated.

Normal Pressure	*Pull of Pan*

FIG. 238.

Graph. On squared paper take two axes Ox and Oy and represent on these, to scale, the loads.

Plot points like P (Fig. 239) where OM represents the weight of the box and its load, and PM the pan weight and its load to scale. All the points like P will be found to lie approximately on a straight line.

By aid of a stretched black thread, determine the straight line lying most evenly amongst the plotted points, and draw this line. From this line find the greatest load that can be placed in the pan so that the box will not move when it contains 270 grammes, and test the result by experiment.

Deductions. In the first experiment with 500 grammes load in the box, at every instant before motion there was equilibrium. For equilibrium the vector polygon must be closed, and therefore the sum of the components of the forces in any direction must be zero. Hence, the friction resisting the motion of the box must have been always equal to the pull of the string on the box. Hence, for a given pressure between the box and plank, the friction may have any value from zero up to a certain maximum value, called the **limiting friction**. If there was no friction at the pulley, the load due to the scale pan and its contents would measure the friction; if there was friction at the pulley, the pull of the string on the box—which measures the friction between the box and the board—would be slightly less than the load. If the straight line goes through the origin, it shews that the pull necessary to turn the pulley is too small to be measurable; if it does not, then the intercept on the y axis gives the friction due to the pulley (OA in Fig. 239).

Draw through A a line parallel to Ox.

Distances measured upwards from this line to the sloping one give to scale the pull on the box. In Fig. 239 PM_1 is the pull on the box for a normal pressure AM_1, and PM_1 therefore measures the limiting friction in this case.

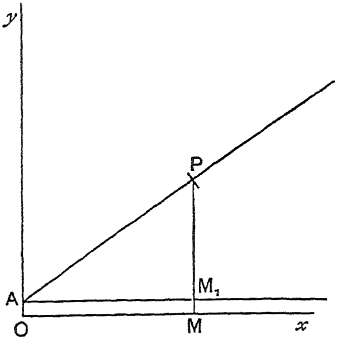

FIG. 239.

Coefficient of Friction. Since AP is a straight line, the ratio $\frac{PM_1}{AM_1}$ or $\frac{\text{limiting friction}}{\text{normal pressure}}$ is always the same, no matter what the pressure. This ratio is called the **coefficient of friction** for the two surfaces, and is always denoted by the letter μ.

EXPT. XIII. Place the box at the narrow end of the board so that it overlaps and the area of contact is less than before; shew that μ is unaltered.

EXPT. XIV. Pin a sheet of paper to the board and shew that μ has a different value from that formerly obtained.

Laws of Statical Friction. The laws thus roughly established are:

(i) *Friction is a passive force, only called into play by the action of other forces; it tends to prevent motion and may have any value from zero up to a certain maximum, depending on the normal pressure and the nature of the surfaces.*

(ii) *Limiting friction is independent of the area of contact.*

(iii) *Limiting friction is dependent on the nature of the surfaces.*

(iv) *Limiting friction is proportional to the normal pressure,* $F = \mu W$.

Friction and Stress. If F is the force of friction on the box, $-F$ is the force on the plank, and friction is therefore of the nature of a stress.

Angle of Friction. Draw PQ (Fig. 240) vertically downwards to represent to scale one of the normal pressures, and QR horizontally for the pull on the box ($-$the limiting friction);

then, since there is equilibrium, the closing line RP must represent in magnitude, direction, and sense, the total reaction of the plank on the box.

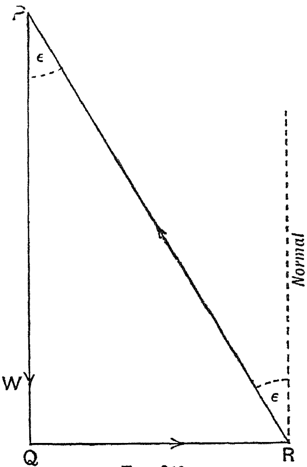

FIG. 240.

The angle ϵ, between RP and the normal to the plank, is called the **angle of friction**, and $\mu = \tan \epsilon$. Measure ϵ and μ. ϵ is always the same, no matter what the pressure, if the box is on the point of moving, for $\frac{F}{W}$ is constant.

When there is no pull on the box, RP is vertically upwards (QP); as the pull increases, RP slopes more and more away from the normal and away from the sense of the attempted motion, until the maximum angle ϵ is reached. Expressed in slightly different words, the total reaction of the surface may be inclined to the normal at any angle between zero and ϵ.

(1) If the box weighs 6 ozs., and it is loaded with 1·5 lbs., and the horizontal pull on it when on the point of motion is 15 ozs., find ϵ and μ.

(2) If the box weighs 10 ozs. and is loaded with 1 lb., and the coefficient of friction is $\frac{1}{3}$, find the pull on the box and the total reaction of the surface of the plank when motion is about to take place.

(3) If the box weighs 15 lbs. and $\mu = \frac{2}{7}$, would equilibrium be possible with a horizontal pull of (i) 3 lbs. weight, (ii) 6 lbs. weight?

(4) A block of stone weighs half a ton and rests on a fixed stone with a horizontal top; a horizontal push of 500 lbs. weight just causes the block to be on the point of motion. What is the angle of friction and what is μ?

(5) A 1000 gramme weight rests on a table; the angle of friction is 25°. What is the least horizontal force that will produce motion?

(6) A cube of side 2″ rests on a horizontal plane. The weight of the cube is 7 lbs., and it is pushed with a horizontal force of 3 lbs. weight without producing motion. The line of action of the force passes through the centre of the cube and is perpendicular to one face. Draw a diagram of the axes of the forces acting on the block (they are concurrent), and find the reaction of the surface.

(7) Draw a diagram of the forces in Ex. 6 when the applied force is $\frac{1}{3}$ of the way up, and find the surface reaction.

Force not Parallel to the Surface. If the string in the friction experiment is not quite horizontal, the normal pressure between the surfaces in contact is no longer the weighted box. Knowing, however, the slope of the string, the normal pressure and the friction can be found.

EXAMPLE. *The string slopes upwards at an angle of 15° with the horizontal, the weight of the box, etc., is 17 lbs., and a pull of 5·5 lbs. weight causes the box to be on the point of motion. Find the angle of friction ϵ, and the frictional force called into play.*

Draw AB (Fig. 241) vertically downwards of length 17 cms., $BC = 5{\cdot}5$ cms. sloping upwards at an angle of 15°. Then **CA** is the total reaction of the plank on the box, and $C\hat{A}B = \epsilon$.

Scale CA, and measure $C\hat{A}B$ by a scale of chords. Set off $AE = 10$ cms. along AB, and draw EF perpendicular to AE; then EF on the 10 cm. scale measures μ $(= \tan \epsilon)$. Look up a table of tangents, and compare ϵ thus determined with the value obtained by the scale of chords.

Draw CD perpendicular to AB; then **BD** and **DC** are the vertical and horizontal components of **BC**; hence the friction on the box is given by **CD**.

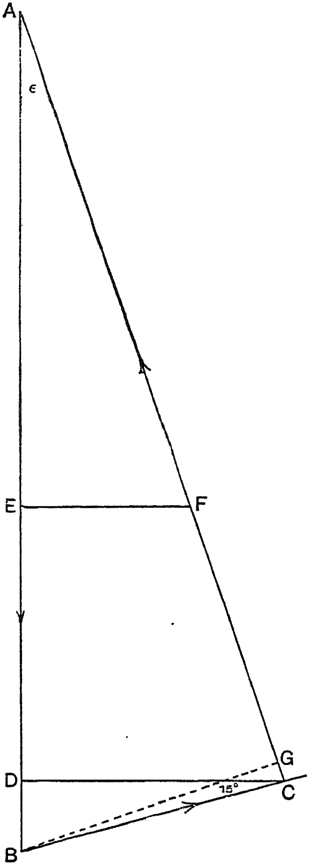

FIG. 241.

BD is the upward pull on the box, due to the string tending to lift it off the plank; hence the normal reaction of the plank is not **BA** but **DA**.

Draw *BG* perpendicular to *AC*. If the inclination of the string had been that of *BG*, the force given by **BG** would have been the pull that would have caused the box to be on the point of motion.

Evidently, this is the least pull possible when the box is in limiting equilibrium. Any force greater than **BG**, if in the same direction, will cause motion; any force less than **BG** will not cause the box to be on the point of motion. Scale **BG** and measure the angle it makes with the horizontal.

(8) The box, etc., weighs 11 lbs.; the coefficient of friction is 0·4. Find the least pull on the box that will cause it to be just on the point of motion, and the amount of friction called into play.

(9) The box, etc., weighs 21·5 lbs.; a pull of 12 lbs. applied at an angle of 20° with the horizontal just causes the box to be on the point of motion. Find the least force that will move the box, and the corresponding normal pressure and friction.

(10) The box weighs 9 lbs. and $\mu=0{\cdot}3$. A horizontal force of 27 lbs. is applied to the box. Is there equilibrium; and if so, what is the angle the total reaction of the plank makes with the vertical?

(11) A harrow weighs 6 cwts., the chains by which a horse can pull it along make 20° with the horizontal. If the horse exerts a pull of 2 cwts. along the chains, find the total reaction of the ground in lbs. wt., the angle it makes with the vertical, and the resistance corresponding to the friction between harrow and ground.

(12) A block weighing 37 lbs. is to be pulled along a horizontal plane by a rope; find the least possible pull and its direction if the coefficient of friction is 0·37.

(13) The least force that will move a chair weighing 10 lbs. along a rough floor is 6 lbs.; find the angle and coefficient of friction and the reaction of the ground.

(14) Two men push a side-board along, the side-board weighs 7·5 cwts. and the angle of friction is 25°. Find the force the men must exert if (*a*) they push horizontally, (*b*) downwards at an angle of 20° with the horizontal, (*c*) upwards at 20° with the horizontal, (*d*) find the direction in which they must push in order that the force they exert may be as small as possible.

(15) A block weighing 17 kilogrammes rests on a rough horizontal table, the angle of friction being 37°. If a horizontal force of 5 kilogrammes weight acts on the block, find the least additional force that will cause motion. What is the greatest horizontal force that can be applied without motion taking place?

The Inclined Plane with Friction.

EXAMPLE 1. *A block rests on a board; the latter is tilted about a horizontal axis through its end. The coefficient of friction being* 0·2, *find the angle at which the block begins to slide and the greatest friction called into play.*

Whilst there is equilibrium, the reaction of the board on the block must be equal to the weight of the block and in the same line (since these are the only two forces acting). The angle between the vertical and the normal to the plane is the same as the angle between the plane and the horizontal, and as the former is equal to ϵ when the block is on the point of sliding the angle of the plane must also be ϵ, and $\tan \epsilon = 0{\cdot}19$. Draw AB (Fig. 242) horizontally of length 10 cms., and BC vertically of length 1·9 cms.; then CAB is the angle of the plane.

Draw any length LM vertically downwards to represent W, LN at an angle ϵ to LM, and MN perpendicular to LN, as indicated in Fig. 242. **MN** is the friction and **NL** the normal reaction. Measure the friction as a decimal of W.

To do this, set off as indicated $MW = ML$, $MI = 5$ inches, and draw through N a line parallel to WI. Measure the intercept on ML on the $\frac{1}{2}''$ scale.

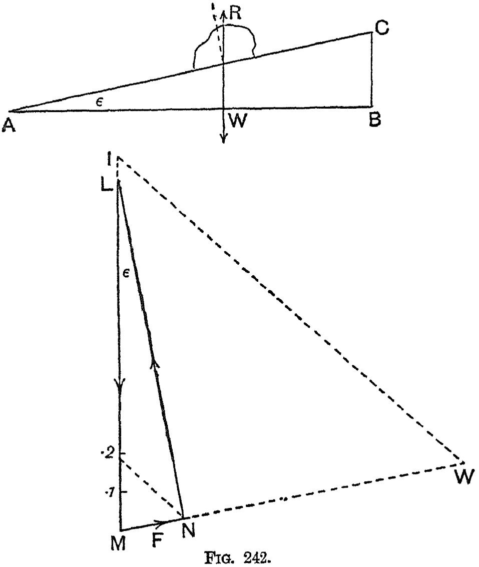

FIG. 242.

EXAMPLE 2. *A block of weight* 10 *lbs. is supported on an inclined plane by a horizontal force. If* $\mu = 0{\cdot}3$, *and the plane rises* 1 *vertically in* 2 *horizontally, find the value of the horizontal force that will just cause the block to be on the point of motion,* (i) *upwards,* (ii) *downwards.*

First draw the plane AC (Fig. 243) by drawing $AB = 4''$ horizontally, and $BC = 2''$ vertically, then a normal MN to the plane. Set off along the normal $MN = 2''$, and (i) $NK = 0{\cdot}6''$ down, and (ii) $NK_1 = 0{\cdot}6''$ up the plane.

When the body is about to move *up* the plane, the total reaction of the surface has the direction MK, when *down* the direction is MK_1.

Next draw the vector polygon, a line PQ vertically downwards 10 cms. long, to represent the weight W of the block, then from the two ends of PQ, a horizontal line QR and one PR parallel to KM. Then **QR** gives the horizontal force which will just cause the block to be on the point of motion up the plane, and **RP** is the total reaction of the surface.

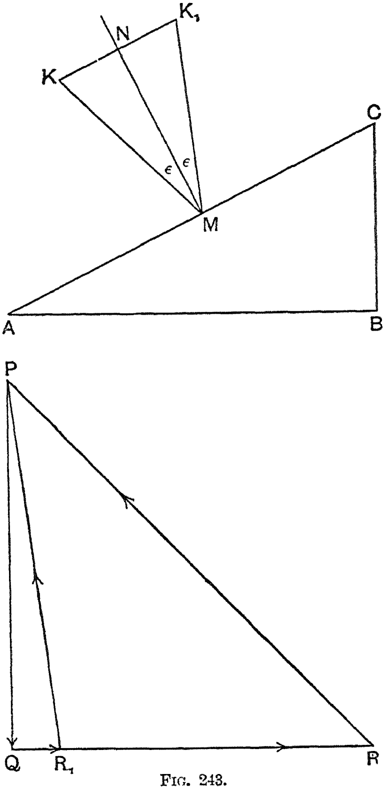

FIG. 243.

Draw PR_1 parallel to MK_1. Then $\mathbf{QR}_1$ measures the horizontal force which will just cause the block to be on the point of motion down the plane, and $\mathbf{R}_1\mathbf{P}$ the corresponding reaction. Would the block rest on the plane if the horizontal force were zero?

EXAMPLE 3. *The problem as before, but the slope of the plane is now* 1 *in* 8.

The graphical work is as in the previous example, but notice now that R_1 comes to the left of W and $\mathbf{QR}_1$ is from right to left, shewing that the pull is to be replaced by a push down.

Notice that whether the force pulls up or pushes down depends on the relative magnitudes of ϵ and α, where ϵ is the angle of friction and α the inclination of the plane.

Minimum Force and Inclined Plane. If $\alpha > \epsilon$ then the body will not rest on the plane without a supporting force. (Why?) This condition being fulfilled, the total reaction **QR** (Fig. 244) of the surface makes an angle of $\alpha - \epsilon$ with the vertical when sliding down is about to take place; hence the least force **RP**, which will prevent motion down the plane, must be perpendicular to RQ or make an angle $\alpha - \epsilon$ with the horizontal, or ϵ with the plane and below it

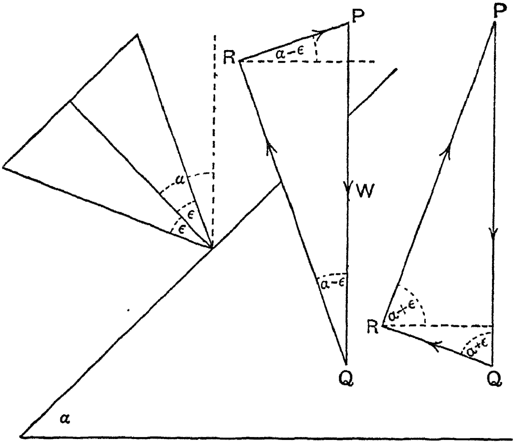

FIG. 244.

When motion is about to take place up the plane, the total reaction makes an angle $\alpha + \epsilon$ with the vertical, and the least force is perpendicular to this reaction and makes an angle $\alpha + \epsilon$ with the horizontal, or ϵ with the plane and above it.

If $\alpha < \epsilon$ (Fig. 245), then, when the body is on the point of motion down the plane, the total reaction makes $\epsilon - \alpha$ with the vertical, and the least force makes $\epsilon - \alpha$ with the horizontal, or ϵ with the plane downwards but above the plane.

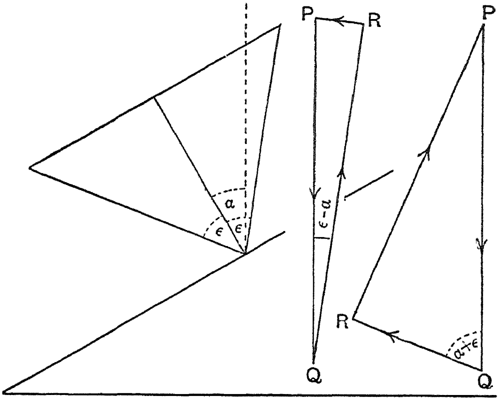

FIG. 245.

When motion is about to take place up the plane the least force makes ϵ with the plane upwards.

(16) A gun has to be dragged up a steep hill (slope 1 in 5); the surface resistance is equivalent to an angle of friction 40°. Find the best angle to which the ropes should be adjusted. If the gun weighs 1 ton find the value of the least force. What force would have to be applied if the ropes were pulled (*a*) parallel to the ground, (*b*) at an angle of 20° with the ground?

(17) A weight of 3 kilogrammes is supported on an inclined plane (rising 1 in 4) by a force parallel to plane. Find the greatest and least values this force can have so that the weight may not move if $\mu = 0{\cdot}2$.

(18) A weight of 7 cwts. is supported on an inclined plane, rising 1 in 1; if $\mu = 0{\cdot}3$ find the least force that can support the weight.

(19) In the last example find the least and greatest forces parallel to the plane so that the weight may be (i) on the point of moving up, (ii) on the point of moving down.

(20) Find the least horizontal force that can move a block weighing 11 lbs. up an inclined plane of inclination 30°, the angle of friction being 15°.

Find also the least force that will prevent motion downwards and cause motion upwards.

What are the values of the friction called into play in the three cases?

(21) Find the force parallel to a plane of inclination 60° that will support a block of weight 15 lbs. on the plane, the coefficient of friction being 1/3. Find also the least force that can move the block up the plane.

(22) A force of 17 lbs. weight will just support a block of weight 40 lbs. on a plane inclined at an angle 55° if applied parallel to the plane. What is the greatest force that can be applied parallel to the plane without causing motion?

(23) What is the inclination of a plane, coefficient of friction $\frac{1}{2}$, if the minimum force necessary to move a block weighing 25 lbs. up it is 15 lbs.? (Remember the direction of the minimum pull is perpendicular to the total reaction of the surface and this makes an angle ϵ with the normal.) For this plane what is the force that will just support the block?

(24) A block of weight 5 cwts. is kept at rest on a rough inclined plane by a rope AB fastened to a point A on the block and to a point on the plane. The plane rises 3 vertically to 5 horizontally and $\mu=0{\cdot}38$. Find the length of the rope AB that will give the least tension if A be 1 foot distant from the plane.

(25) Two light rods are pin-jointed together and rest in a vertical plane on a rough board ($\mu=0{\cdot}4$). A weight $W=9$ lbs. is suspended from the joint; find the greatest angle between the rods consistent with equilibrium.

(26) In the example on p. 156, if the coefficient of sliding friction for the piston be $\frac{1}{5}$, find the total reaction of the guides and the force transmitted along the connecting rod. Find also the tangential force urging the crank forward for the various positions given.

Harder Problems on Friction. In some cases a little ingenuity is necessary to effect the graphical construction.

EXAMPLE. *A uniform beam, of length* 17 *ft. and weight* 3 *cwts., rests against a smooth vertical wall and a rough floor for which the coefficient of friction is* 0·4. *Find the position of the beam when it is just on the point of slipping down, and the friction which prevents the motion.*

Notice first that the reaction at A is horizontal, and that at B inclined at ϵ to the vertical. First draw the vector polygon, XY (Fig. 247) vertically down, of length 15 cms.; YZ horizontally, of length 6 cms.; and join XZ. Join Z to the mid-point M of XY.

From any point A draw AB parallel to MZ (Fig. 246) and of length 1·7″, AC vertical and CB horizontal. Then AB gives the position of the beam relative to the wall AC and the ground CB.

***Proof.** Since $\frac{YZ}{XY}=\frac{6}{15}=0{\cdot}4$, and YZ is horizontal, ZX gives the direction of the reaction of the ground at B, and XYZ must be the vector triangle for the forces on the beam.

The three forces on the beam, viz. the weight through G (the mid-point) and the reaction at A and B, must pass through a point O. Produce OG to K (as in Fig. 246), then OKB and XYZ are similar; $\therefore \frac{OK}{KB}=\frac{XY}{YZ}$; if, then, XY be bisected at M, $\frac{MY}{YZ}=\frac{GK}{KB}$, and therefore MZ is parallel to AB.

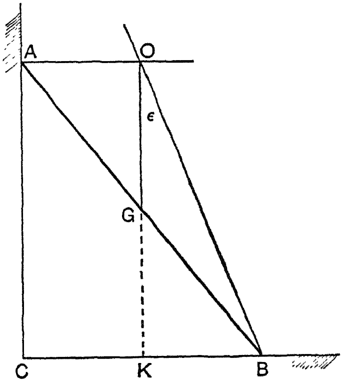

FIG. 246.

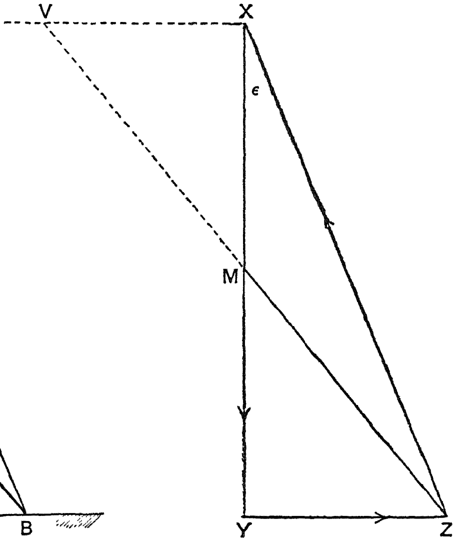

FIG. 247.

A Simpler Proof. Replace the uniform heavy beam by a light rod having 1·5 cwts. concentrated at its ends; then for the equilibrium at A we have the load given by **MY**, the reaction at A given by **YZ**, and therefore the push of the beam along BA must be parallel to ZM the closing line of the vector polygon for A. YZ is known because it equals 0·4 XY.

Similarly, at B we have the load **XM**, the ground reaction **ZX** and the push **MZ** of the beam along AB.

EXAMPLE. *A uniform ladder rests against a wall at an angle of* 30°. *If it be just on the point of slipping down, and the angle of friction is the same for wall and ground, find the coefficient of friction.*

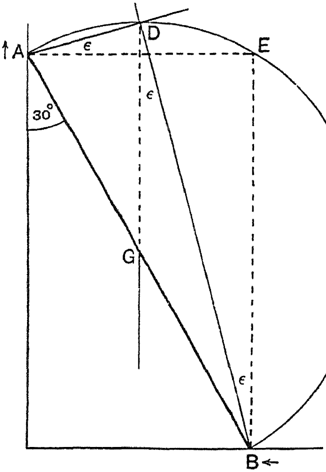

FIG. 248.

AB (Fig. 248) represents the ladder and *G* its M.C. through which its weight is supposed to act. Then, since *B* is on the point of moving to the right, the friction acts from right to left and the total reaction at *B* makes some (unknown) angle with the normal, and slopes towards *A*. At *A*, the total reaction slopes upwards, for the friction acts upwards. Since the angle of friction is the same at *A* and at *B*, these reactions must intersect at right angles; the point *D* of intersection, therefore, lies on a semicircle having *AB* as base. For equilibrium, the vertical through *G* must be concurrent with the reactions, hence:

Describe a semicircle on *AB*, draw the vertical through *G* to intersect it at *D*, draw *AD* and *BD* the reaction lines, and measure μ ($= \tan \epsilon$).

(27) As in previous example, only the M.C. of the ladder is $\frac{1}{4}$ of the way up from the bottom.

(28) Find where the M.C. of the ladder must be if the coefficients of friction for the wall and ground are 0·3 and 0·5 respectively.

(29) A uniform bar AB of weight 27 lbs. rests on rough ground at A, and against a smooth bar at C. The inclination of the bar is 30° to the horizontal; $AB=8$ ft., $AC=5$ ft.; find the reaction of the ground and the coefficient of friction if the bar is about to slip.

EXAMPLE. *The coefficients of friction for a ladder resting against a wall and on the ground are* 0·3 *and* 0·6. *Find the limiting position of the ladder supposed uniform, and the friction on the wall, if the ladder weighs* 120 *lbs.*

Draw a vertical line PQ (Fig. 249) of length 12 cms. to represent the weight, and from the ends draw PR and QR parallel to the reactions of the wall and ground. Bisect PQ at S and join RS; then RS is the direction of the ladder.

Proof. Replace the beam by a light rod with equal loads at the ends.

In Fig. 249 SQ is the load at the bottom, QR the reaction of the ground, and, therefore, RS must give the push of the beam on the ground. Hence RS is parallel to the beam.

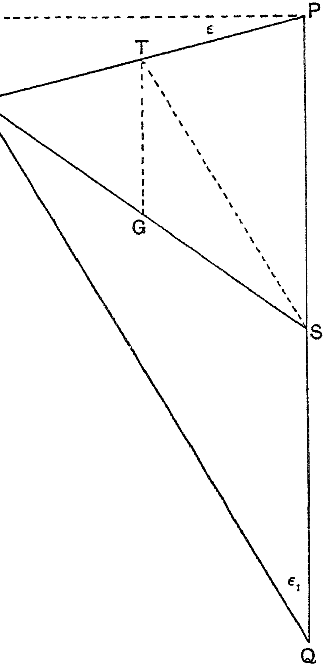

FIG. 249.

(30) Shew, by drawing ST parallel to RQ, and TG parallel to PQ, that RS will represent the beam position. Notice that the reactions RT and TS are in the right directions, and are concurrent with the vertical through G the mid-point of RS.

* Beam on Two Rough Inclined Planes.

EXAMPLE. *A beam rests on two planes of inclinations 30° and 45° for which the coefficients of friction are 0·12 and 0·2. Find the two positions of the beam when in limiting equilibrium, the mass-centre of the beam being $\frac{3}{7}$ of its length from the 30° plane. If the beam weigh 700 kilogrammes, find the friction on the planes in the two cases.*

(The vertical plane of the beam is supposed to intersect the planes in their lines of greatest slope.)

Suppose the 30° plane to be on the left.

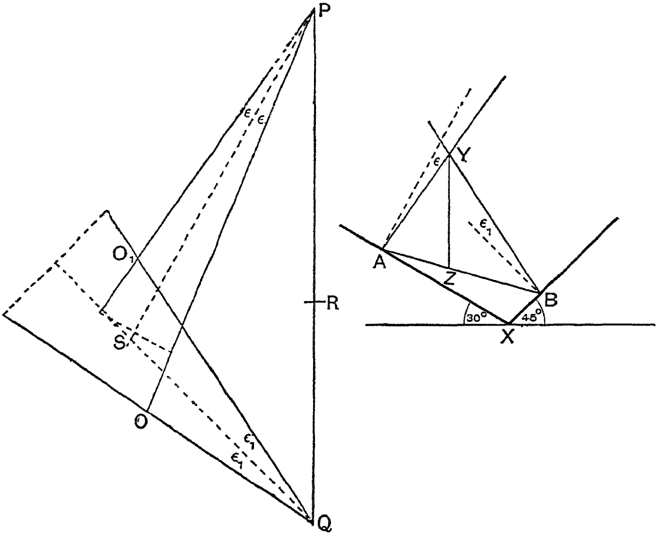

FIG. 250.

Set off PQ (Fig. 250) downwards of length 7″ and draw PS and QS parallel to the normals to the planes, *i.e.* PS making 30° and QS making 45° with the vertical. Mark the point R where $PR = 4''$ and $QR = 3''$. Then set off the friction angle ϵ, where $\tan \epsilon = 0{\cdot}12$, on both sides of PS, and, similarly, set off ϵ_1 on both sides of QS.

When the beam is about to slide down the 45° plane the reaction of the plane tends to prevent the sliding and is, therefore, to the right of the normal; at the same time the reaction of the 30° plane tends to prevent sliding up and is, therefore, to the right of its normal. For this case then QO_1 and PO_1 (Fig. 250) are the directions of the reaction lines.

When the beam is about to slide down the 30° plane the reaction lines will be parallel to PO and QO for the 30° and the 45° planes respectively.

Now suppose the beam replaced by a light rod having 400 kilogrammes concentrated at its end A in contact with the 30° plane, and 300 kilogrammes at the other end B. Then for the equilibrium at A we have **PR** ($=400$ lbs.), the reaction of the 30° plane and the push of the rod.

When A is about to slide up, the reaction of the plane is $\mathbf{O_1P}$, and hence $\mathbf{RO_1}$ gives the push of the rod at A; therefore the rod must be parallel to O_1R.

Similarly, when A is about to slide up, **PO** is the reaction and **RO** must give the push of the beam at A; hence the beam must now be parallel to OR.

Draw the planes XA of 30° inclination to the left, and XB of 45° inclination to the right. At any point A on the former, draw AY parallel to PO_1, and AB parallel to O_1R. From B on the plane XB draw BY parallel to QO_1. Then through Y, the point of intersection of AY and BY, draw a vertical cutting AB in Z. See that $AZ/ZB=\frac{3}{4}$.

In a similar manner, draw the other limiting position of the beam A_1B_1, and verify the accuracy of the vector polygon construction again.

To determine the friction in the first case, draw O_1F perpendicular to PS; then **FP** is the normal reaction and $\mathbf{O_1F}$ the friction at A.

It is evident that there are not always two positions of limiting equilibrium, *e.g.* if RO is steeper than 45°, or RO_1 steeper than 30°, there will only be one position; if both happen together there is no position of limiting equilibrium.

(31) A ladder rests against a vertical wall. The angles of friction for the wall and ground with the ladder are 20° and 40°. Find the position of the ladder when just on the point of slipping down, if the position of the M.C. is $\frac{1}{3}$ up the ladder from the ground.

(32) A heavy beam weighing 1000 lbs. rests in limiting equilibrium with one end on the ground and the other on a plane of inclination 30°. If the coefficient of friction $=0 \cdot 4$ for both ends, and the beam be about to slip when inclined at 20° to the horizontal, find the position of the M.C.

(33) As in previous exercise, only μ is 0·4 for the ground and 0·3 for the plane.

(34) A heavy uniform beam *AB* weighing 700 lbs. rests with its end *A* on a plane of inclination 30° and coefficient of friction 0·3. The other end *B* is on a plane of inclination 40° and coefficient of friction 0·4. If the end *B* is about to slide down when the beam is horizontal, find the position of its M.C.

(35) A uniform rod of length 7″ rests inside a vertical rough hoop of radius 5″. It is found that the greatest inclination that the rod can have to the horizontal is 30°. Find the coefficient of friction.

(If *O* is the centre of the hoop and *AB* the rod inclined at 30° to horizon, draw the circle circumscribing *OAB*; this cuts the vertical through *M* (the mid-point of *AB*) in *C*; then *CA* and *CB* are the reaction lines at *A* and *B*. Measure the tangent of the angle between each of these lines and the corresponding radius.)

(36) In the previous exercise if the coefficient of friction at the ends are 0·3 and 0·2 (lower and upper ends), find the position of the M.C.

(37) The angle of friction being 20° at each end, and the rod uniform, find the position of the rod in the loop when on the point of slipping down.

Draw *AB* in any position in the circle. Join *OA* and *OB* and draw the reaction lines at *A* and *B*. Join the point of intersection of these lines with *M* the mid-point of *AB*; this last line represents the vertical. Measure the angle between it and *AB*; this gives the position of the beam.

(38) The angle of friction being 25° at each end, find the limiting position of the rod when the mass-centre is distant 2″ from the upper end of the rod.

(39) A heavy beam weighing 1050 lbs. rests in limiting equilibrium with one end on the ground and the other on a plane of inclination 60°. If the coefficient of friction is 0·4 for both ends, and the M.C. is $\frac{2}{5}$ up from the ground, determine the position of equilibrium and the frictions at the ends.

(40) As in previous case, but the coefficient of friction for the end in contact with the inclined plane is 0·25.

(41) A heavy uniform beam weighing 570 lbs. rests with one end *A* on a plane of inclination 30° and the coefficient of friction 0·2. The other end *B* is on a plane of inclination 50° and the coefficient of friction is 0·4. If the end *B* is about to slip down, find the position of the beam and the friction on the two planes.

Find if another position of limiting equilibrium is possible.

EXAMPLE. *Fig.* 251 *represents part of an ordinary bicycle screw-spanner. By means of the screw thread* S *and the rack* BO_1, *an upward force* α *is given to the movable piece. If a short rod* DE *is placed between the jaws, required to find the force* β *which is exerted on the rod when the magnitude of* α *is* 3 *lbs. weight, and the coefficient of* *friction between the movable and fixed parts is* 0·4. *The distance between the axes of* α *and* β *is* 1″, *between* A *and* G 1·5″, *between* B *and* G 0·6″.

The effect of $-\beta$ downwards on the movable piece, is to press the latter against the fixed part at A and B, and since the lower jaw is tending to move upwards the friction acts downwards, and hence the total reactions at A and B slope as in Fig. 251.

Draw the axes of the four forces, α, β and the two reactions, the distances being taken double the actual ones. Find the points of intersection O and O_1 of β and the reaction at B, and α and the reaction at A. For equilibrium, the resultant of α and the reaction at A must balance β and the reaction at B, hence this resultant must have the direction OO_1.

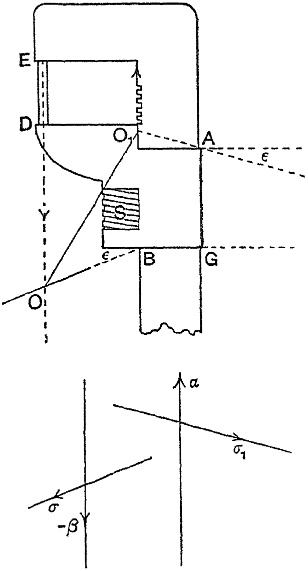

Fig. 251.

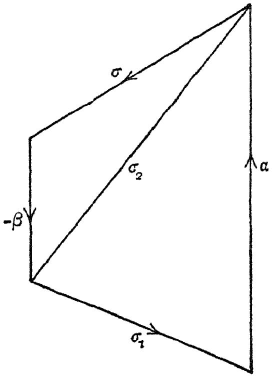

Fig. 252.

Draw α vertically upwards of length 3″, and through its end points draw σ_1 and σ_2 parallel to AO_1 and OO_1 respectively. This gives σ_1 the reaction at A. From the ends of σ_2 draw σ and $-\beta$ parallel to OB and P respectively. Then $\alpha+\sigma-\beta+\sigma_1=0$ and β is the force of the moving piece on the rod.

Another, and perhaps slightly simpler, way of solving the problem is as follows. Find the point F of intersection of the reactions at A and B. Resolve α acting along BO_1 into two parallel forces along ED and the parallel through F. The former component is the reaction of the movable piece on the rod.

Compare the results obtained by the two methods.

EXAMPLE. *Fig.* 253 *represents the load stage and part of the rack of a screw jack for raising loads eccentrically.* AB *is the pitch line of the rack along which the lifting force* α *acts, the load* β *is carried by* C. D *and* E *are parts of the casing against which the rack presses when a load is being raised.* AD = 0·5″, DE = 5″ *and the distance of the load line from* AB *is* 3·5″. *When the load is* 2·6 *cwts., find the smallest magnitude of* α *necessary to raise the load if the coefficient of friction between the rack and casing is* 0·3.

Draw the axes of the forces α and β and the reaction lines of D and E. Find the point F of intersection of the two latter, and resolve β through C into two parallel forces, one through the point F and the other along AB. The component along AB is α.

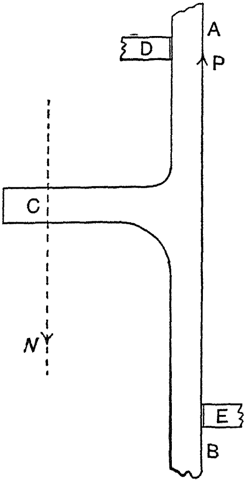

FIG. 253.

(42) Find α also by the method first given for the previous example and compare the results.

(43) Find the least value of α necessary to prevent the load descending.

EXAMPLE. ABCD (Fig. 254) *represents a horizontal drawer. It is attempted to pull the drawer out of its case by a non-central handle* P. *Neglecting the friction at the bottom of the drawer, how far may the drawer be pulled out before jamming, if* $\mu=0·6$?

Suppose the drawer is pulled out a distance AA_1; then, owing to the pull being non-central, the drawer is pressed at C against the right slide, and at A_1 to the left. The reaction at C makes angle ϵ with CD ($\tan\epsilon = 0{\cdot}6$: actually ϵ for motion is a little less than ϵ for rest); the axis of the pull meets this reaction line at O. At A_1, the reaction is along A_1K, making ϵ with A_1B_1. Neglecting the weight of the drawer there are three forces, and three only, acting on it, viz. the pull and the two reactions. For equilibrium these must pass through at point.

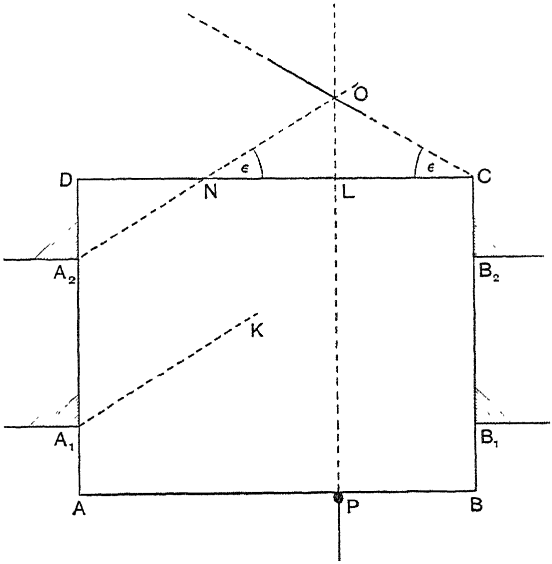

FIG. 254.

Now O is fixed relatively to the drawer, hence A_1 moves until A_1K passes through O, *i.e.* the drawer can be pulled out a distance AA_2, where OA_2 makes an angle ϵ with CD.

Hence to find point A_2: Divide CL into 10 equal parts, set off $LO = 6$ of these. Mark N on CD, where $LN = LC$; join ON, cutting AD at A_2; then the drawer can be pulled out a distance AA_2.

(44) Find the farthest distance that the handle may be from the centre line in order that the drawer may be pulled out to within a quarter of its length, $\mu = 0{\cdot}6$.

(45) The handle being midway between the centre line and a side, find μ if the drawer jams when pulled out half its length.

EXAMPLE. ABC (Fig. 255) *represents a wedge,* DE *a uniform iron rod of weight* 7·2 *lbs., hinged at* D *and resting on the wedge at* E. *The coefficient of friction between the rod and wedge is* 0·3, *and between the wedge and ground it is* 0·4. *The wedge is pushed by a horizontal force* a *so that it is just on the point of motion; determine* a *if the weight of the wedge may be neglected; given*

$$DE = 4{\cdot}6\ ft.,\quad DB = 1{\cdot}9\ ft.,\quad AC = 3{\cdot}2\ ft.,\quad BC = 4\ ft.$$

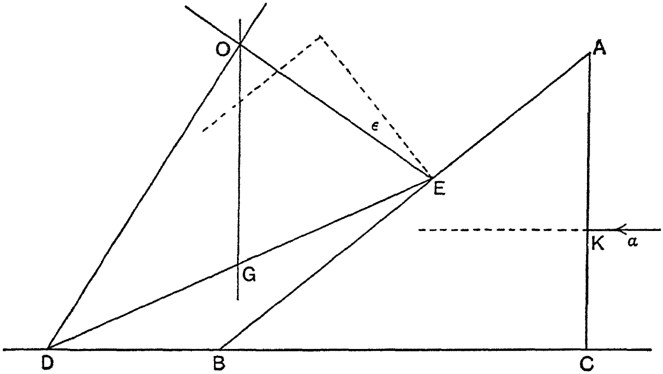

FIG. 255.

The line of the reaction at E is known, also the weight of the rod acts through its mid-point, and since the rod is in equilibrium the direction of the reaction at D is known; hence find the forces by the vector polygon. The force at E on the wedge is now known, the axis of a is known, and also the angle the reaction of the plane makes with the vertical; hence draw the vector polygon for the wedge, and determine a from it.

(46) The weight of the wedge being 3 lbs., acting through the M.C. of the triangle ABC, determine a.

(47) Determine a when $AC = BC = 1$ ft.; μ is the same for both surfaces, and the weight of the wedge is 4 lbs.

In the problems on friction so far discussed it has been supposed that the body considered remains in equilibrium until the total reaction of the surface makes the angle of friction with the normal. This is not always possible, as the body may begin to turn about some point or edge before sliding commences.

EXAMPLE. *A cube of 2″ side rests on a rough horizontal plane, for which* $\mu = 0{\cdot}6$; *it is acted on by a horizontal force perpendicular to a face and passing through the mid-point of the top face. Shew that, however large this force, the cube will not slide, but that equilibrium will be broken by the cube turning about an edge.*

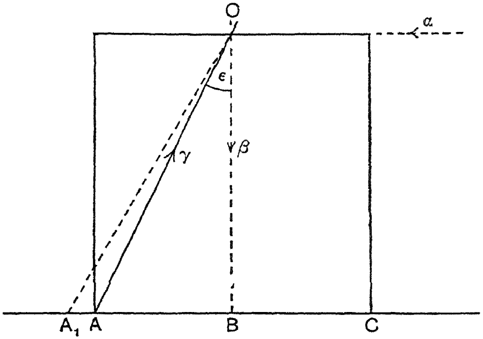

FIG. 256.

Draw a square of 2″ side to represent the central section of the cube containing the axis of the horizontal force.

While the cube is in equilibrium the three forces—horizontal push, weight, and reaction of the plane—pass through a point. This point O, Fig. 256, is determined by the intersection of the horizontal axis and the vertical through the M.C. of the cube. If about to slide, the reaction of the surface makes an angle with the vertical whose tangent is $0{\cdot}6$; draw this line OA_1 through O; it cuts the plane outside the base of the cube. But the plane reaction must act within the base, and hence it is impossible for the cube to be on the point of sliding.

As the push a increases from zero, the total reaction of the surface makes a larger and larger angle with the normal until it comes to the position OA. Evidently when the reaction is at A, AC must barely touch the ground except at A, and the cube must be on the point of rotating about the edge through A.

Draw the vector polygon of the forces when OA is the line of reaction of the ground, and determine the greatest value of α consistent with equilibrium.

EXAMPLE 2. *A cable reel has an outside diameter of* 4 *ft.; the radius, from which the cable is uncoiled, is at a certain moment* 1·36 *ft. The reel is placed on the ground (coefficient of friction* = 0·28*); find the point at which the cable must be taken off, and the direction of the cable at that point, so that the reel may be just on the point of slipping, and find the force necessary to effect this if the reel and cable weigh* 0·32 *tons.*

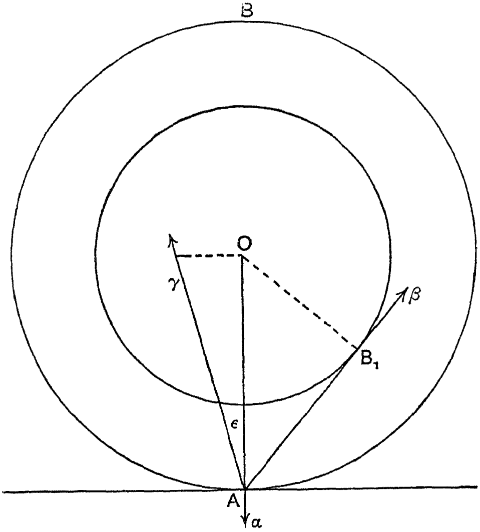

FIG. 257.

Draw (Fig. 257) to scale circles representing the reel AB which rests on the ground, and the layer from which the cable is being uncoiled. Then, since both the weight and the reaction of the ground must act through A, so must the pull of the cable. Draw AB_1 a tangent to inner circle, which gives the direction in which the cable is taken off, and the point B_1 (there are two such points) at which the cable leaves. The vector polygon must now be drawn, the three sides being parallel to α, β and γ respectively; ϵ is the angle of friction.

Measure the pull of the cable and the reaction of the ground.

Since for equilibrium the three forces must pass through A if the cable be taken off at any other point than B_1 the reel will roll.

Try an experiment, illustrating this, with a reel of cotton.

EXAMPLE. *The ground slopes at an angle of 30° and the reel is just on the point of sliding down, find the point at which the cable leaves the reel and the direction and magnitude of the pull on it.*

Draw (Fig. 258) the plane and reel in position as shewn, also the vertical through the centre G of the reel. Since the reel is by supposition about to slide down the friction must act upwards; draw then, at A, a line making the angle of friction with the normal. This line cuts the vertical through G at a point O; from O draw tangents to the inner circle touching it at points B and B_2; these points of contact are the points at which the cable may leave the reel, and the tangents themselves are the directions of the cables.

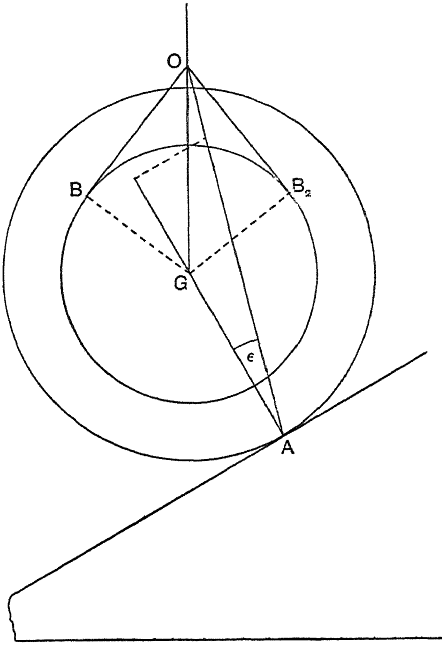

FIG. 258.

Now draw the vector polygons for the two cases and determine the senses of the pulls and their magnitudes.

(48) One end of the cotton on a large reel is fixed to a vertical rough wall; if the reel rests in equilibrium against the wall and is just on the point of slipping down, find the point at which the cotton must leave the reel; given $\mu=0{\cdot}6$, the outer radius of reel $=1''$, the inner radius at which the cotton unrolls $=0{\cdot}45''$.

If the reel weigh 7 ozs. find the tension in the cotton.

(49) One end of the cotton is fastened to a rough plane, $\mu=0{\cdot}3$, of inclination 45°. Find the inclination of the cotton if the reel is about to slide down the plane.

(50) Why is it not possible to cause the reel to be about to slide up the plane by pulling at the cotton.

(51) Stand a thick book upright on a table and push it, perpendicular to the cover, with a pencil, first near the table and gradually increasing the distance until the book topples over. Measure the thickness of the book and the height of the push and calculate μ.

(52) Draw a rectangle of height $4''$ and base $2''$ to represent the right section of a cuboid through its centre, at points distant $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, ... inches from the base; suppose horizontal forces applied in turn until equilibrium is broken. Mark the corresponding points on the base where the total reaction of the surface cuts it, the coefficient of friction being $0{\cdot}3$. Find the highest point at which the force may be applied.

(53) Draw a square of side $3''$ to represent the section of a cube through its centre, and a line through a top corner inclined at an angle of 30° with the horizontal, to represent the line of action of a pull on the cube. The coefficient of friction being $0{\cdot}4$, find the greatest possible pull, if the equilibrium remains unbroken. Draw the axis of the total reaction of the surface. Weight $=10{\cdot}7$ lbs.

(54) In the last example, if the pull be exerted at an angle below the horizontal, find the greatest angle for which sliding is possible and the greatest pull possible if the cube does not move.

(55) ABC (Fig. 259) is the right central section of a triangular prism, ABC being a right angle. BC rests on rough horizontal ground. A rope is fastened to A and pulled horizontally as indicated parallel to BC. If $AB=4{\cdot}1'$, $BC=3{\cdot}5'$, and $\mu=0{\cdot}3$, determine if sliding is possible. If so, find the least value of a that will cause sliding.

(56) If a be reversed in sense and $\mu=0{\cdot}4$ will sliding be possible? Give reasons. What value of a will cause the equilibrium to be broken in this case, and what value of μ would cause it to be broken by sliding with this value of a?

(57) BC (Fig. 260) is part of an inclined plane of inclination 20°. $ABCD$ is a cube kept in position by a string parallel to BC and fastened to D. What is the greatest value of μ consistent with equilibrium? If μ has this value find the greatest value of the pull in the string consistent with equilibrium if the cube weighs 11 lbs.

(58) As in last example, but let string slope upward at an angle of 45° with horizontal.

(59) In Fig. 260 make $AB=2BC$. If $\mu=0{\cdot}3$, find the direction of the string attached to D so that the prism may be on the point of turning about the edge at C.

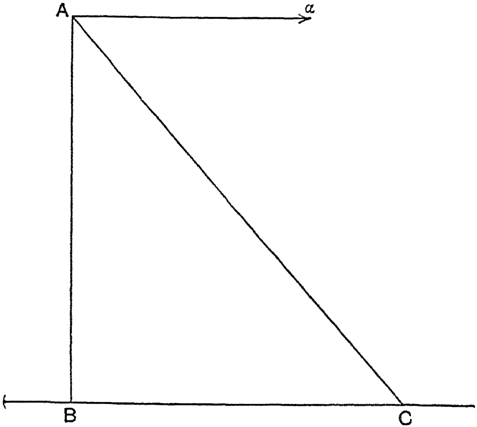

FIG. 259.

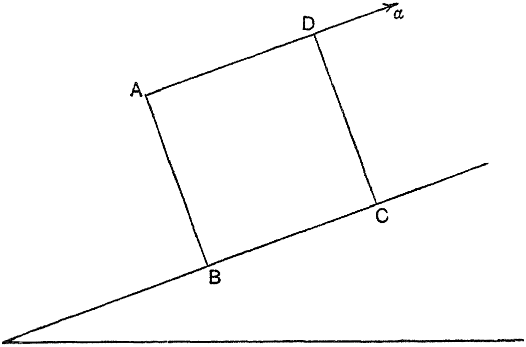

FIG. 260.

MISCELLANEOUS EXAMPLES. VII.

1. An experiment was performed in which a loaded slider was, by a suitable horizontal force P, caused to be just on the point of motion.

Plot the values of P and the weight of the slider W given in the table on squared paper, and determine approximately the value of the coefficient of friction.

P in lbs. wt. -	0·45	0·65	0·84	1·1	1·3	1·4	1·7
W in lbs. wt. -	2·3	3·3	4·3	5·3	6·3	7·3	8·3

2. In Fig. 261 the circles represent the coupled driving wheels of a railway engine. If the engine is starting, show roughly the direction of the pressure between the wheels and the rail. Give a reason for your answer.
(Naval Cadets.)

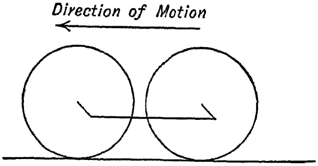

FIG. 261.

3. A particle whose weight is 10 lbs. is placed on a rough plane inclined at an angle of 30° to the horizon; it is acted on by a force up the plane equal to the weight of 6 lbs., acting along the plane; the particle does not move; find the friction between the particle and the plane.

If the particle is just on the point of sliding, find the coefficient of friction. (B. of E., II.)

4. A uniform rod rests with one end against a smooth vertical wall, and the other end on a rough horizontal plane; it can just stand without sliding when its inclination to the horizon is 45°; find the coefficient of friction; also find the inclination when the friction called into play is one-half of the limiting friction. (B. of E., II.)

5. If the angle of friction for an inclined plane be 45°, determine completely the least force that will drag a weight of 100 lbs. down a plane inclined at 30° to the horizontal. (B. of E., II., 1906.)

6. Find graphically the magnitude of the least horizontal force which will support a weight W on a rough plane whose inclination $a < \tan^{-1}\mu$.
(B.Sc., 1904.)

7. Define the coefficient and the angle of friction. A body weighing 500 lbs. is sustained on a rough inclined plane (base twice the height) by a rope pulled in a horizontal direction. Prove that the greatest and least tensions of this rope consistent with equilibrium are about 389 and 134 lbs. wt. (Inter. Sci., 1904.)

8. Find, graphically by preference, the direction in which a force of given magnitude must act if it is just able to move a body of given weight up a rough inclined plane, the coefficient of friction being known.

Shew that when motion is possible there are in general two such directions. (Inter. Sci., 1900.)

9. A beam rests against a smooth vertical wall and a rough inclined plane of inclination a passing through the foot of the wall. Determine the greatest angle the beam can make with the vertical.
(Inter. B.Sc. (Eng.), 1905.)

10. Define friction and limiting friction. Explain briefly what is meant when friction is said to be a passive force.

AB is a uniform rod of weight 10 lbs.; it lies on a rough horizontal table, and is pulled at the end B in the direction of its length by a force of 2 lbs. If AB stays at rest, how much friction is called into play?

Everything being as it was, a thread is tied to the end B and is pulled vertically upwards by a gradually increasing force P; find the least coefficient of friction for which P will begin to lift the point B. How will the rod begin to move if the coefficient of friction equals 0·25? (B. of E., II.)

11. A body is placed on an inclined plane and the coefficient of friction is $\frac{1}{3}$; it is acted on by a force along a line of greatest slope; find the force when it is on the point of making the body slide up the plane.
(B. of E., II., 1903.)

12. A ladder AB rests on the ground at A and against a vertical wall at B. If AB is inclined to the vertical at an angle less than the angle of friction between ladder and ground, shew geometrically that no load, however great, suspended from any point in the ladder will cause it to slip.
(B.Sc., 1905.)

13. A weight rests on a rough inclined plane, whose inclination (α) exceeds the angle (λ) of friction, being prevented from sliding by a force P. Find (geometrically or otherwise) the direction and magnitude of the least force which will suffice for this purpose. (Inter. Sci., 1906.)

14. A uniform circular hoop is weighted at a point of the circumference with a mass equal to its own. Prove that the hoop can hang from a rough peg with any point of its circumference in contact with the peg, provided the angle of friction exceeds 30°.

(Relative to the point of support the M.C. of the hoop and particle lies on a circle of radius half that of the hoop.) (Inter. Sci., 1905.)

15. Draw a horizontal line ABC, $AB=1''$ and $BC=3''$. Let ABC denote a uniform beam of weight w resting on a rough prop at B, and underneath a rough prop at A. Find the direction and magnitude of the least force applied at the end C which will just begin to draw out the beam from between the props. (B. of E., II., 1906.)

(Draw the reaction lines at the points A and B to intersect in D, resolve the weight of the beam acting at its M.C. into two, one passing through D and the other through C, the latter component to be the least possible.)

16. Prove that a sash window of height a, counter-balanced by weights, cannot be raised or lowered by a vertical force, unless it is applied within a middle distance $a \cot \phi$ (ϕ the angle of friction).

If the cord of a counter-balance breaks, the window will fall unless the width is greater than $a \cot \phi$. (B.Sc., 1902.)

17. A square window sash weighing 30 lbs. slides vertically in grooves. From the two upper corners sash cords are carried over pulleys and carry two counterpoises each of 15 lbs. Shew in a diagram the forces acting on the sash when one of the sash cords breaks, and find the least coefficient of friction between sash and grooves that will keep the sash from sliding down, if all other friction may be neglected. (C.S., Div. I., 1905.)

18. If a body having a flat base is placed on a rough inclined plane of inclination i and angle of friction λ, and the body is pulled by a horizontal force P, prove that for equilibrium P must lie between the values $W \tan(i+\lambda)$ and $W \tan(i-\lambda)$ where λ = weight of the body. If $\lambda > i$, explain the second case. (Inter. Sci., 1906.)

19. Define the coefficient of friction and the angle of friction for two rough bodies. A mass of 500 lbs. on a rough inclined plane for which the coefficient of friction is $\frac{1}{5}$ and whose inclination is $\tan^{-1} \frac{1}{2}$ is sustained by a rope which is pulled in a horizontal direction; prove that the greatest and least tensions of this rope are about 389 and 136·4 lbs. wt. respectively.
(Inter. Sci., 1904.)

CHAPTER VIII.

MOMENTS.

To obtain a real grasp of the theory of moments the experiments described in Appendix I. should be performed. It is only by actually performing such experiments that the physical meaning of **turning moment** or **torque** becomes realised.

DEFINITION. **The moment of a force about a point is the product of the force and the perpendicular distance of its axis from the point. It is positive if the direction and sense of the force relative to the point is contraclockwise, negative if clockwise.**

Geometrical Representation. From the point draw any line to the axis, then the area of the parallelogram which has this line and the vector of the force as adjacent sides, measures the moment. If the sense of the boundary, as given by the vector, is contraclockwise, the moment is positive.

Thus, if O (Fig. 262) is the point, a the axis, and α the vector of the force, the positive area $OABC$ measures the moment.

Whatever the direction of OA, the area and the sense of the boundary remain the same.

For a given force, the moment in general changes when the point is changed in position. In Fig. 262 the farther O is to the left of a the greater the positive moment. When O is on a the moment vanishes, and on O crossing to the right of AB, the moment is negative. *If, however,* O *moves on a line parallel to* a, *the moment remains unaltered.*

It is important to notice that when the moment of a force is zero about a point *O*, we may have either *OA* or *AB* zero, *i.e.*, the force itself may be zero, or it may pass through the point *O*; to decide which alternative is correct further information is necessary.

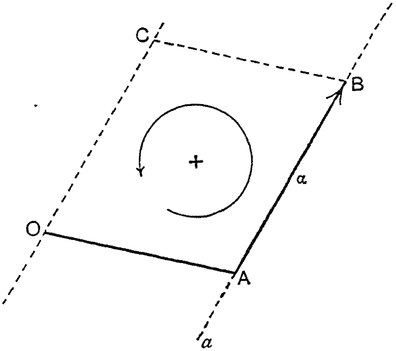

FIG. 262.

Taking account of sense, *OABC* or an area equivalent to it is called the **momental area** of the force *a* in *a* about the point *O*.

Unit Moment. There is no special name in general use for the unit moment. If we use a lb. wt. as the unit of force and a ft. as the unit of length, then the unit moment may be called *one lb. ft.* moment. This of course might mean a force of 1 lb. wt. at a ft. distance, or 2 lbs. wt. at 6″ distance, etc.; later we shall see that these are really equivalent. Whatever the units of force and length used, it is necessary to specify them in giving a number as the measure of a moment.

Graphical Measurement of a Moment.

EXAMPLE. ab (Fig. 263) *is the axis,* **AB** *the vector of a force. Required the measure of the moment of the force about a point* O (*distant* p *from* ab). AB = 2·65″, *scale* 1 *cm. to* 1 *lb. wt., and* p = 4·36″.

Take a pole P of the vector polygon at a distance h (2″) from AB. Through O draw a line A_1B_1 parallel to ab, and through any point Q in ab draw QA_1 parallel to PA, and QB_1 parallel to PB, cutting A_1B_1 in A_1 and B_1.

Measure A_1B_1 on the $\frac{1}{2}$ cm. scale; it is the moment of the force **AB** about O in lbs. inches.

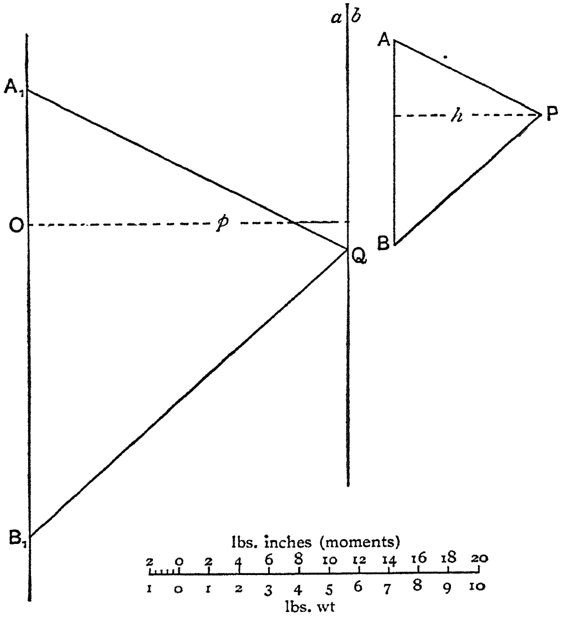
a
b
A
A₁
h
P
O
p
Q
B
B₁
lbs. inches (moments)
2 0 2 4 6 8 10 12 14 16 18 20
1 0 1 2 3 4 5 6 7 8 9 10
lbs. wt

FIG. 263.

Proof. PAB and QA_1B_1 are similar triangles, and therefore

$$\frac{p}{A_1B_1}=\frac{h}{AB};$$

$$\therefore\ 4{\cdot}36AB=2A_1B_1.$$

From the ratios it is seen that p and h being in inches, A_1B_1 must be measured on half the AB scale.

(This is really our old argument of p. 46; the moment is represented by a rectangle $p \cdot AB$, or by an equal rectangle A_1B_1h. If the base h is the unit of length then the altitude A_1B_1 measures the area; if h be twice the unit of length, the altitude is only ½ what it was before, and to obtain the old altitude we must multiply by 2, or use a scale with ½ the old unit.)

Sense of a Moment. If the vector polygon and moment diagram be drawn according to rule, an inspection of the latter will shew whether the moment is positive or negative.

The radial lines of the vector polygon are always supposed drawn in the order of the vectors, and the link polygon lines in the same order.

Hence the order in which the points A_1, B_1, ... are drawn gives the sense of the intercept—downwards in the case considered.

The force being in ab and downwards, the moment about O is negative; hence the rule: *if the intercept has a downward sense the moment is negative, if upwards, a positive sense.*

(1) Verify this rule by taking O on the other side of ab.

(2) Verify again by taking P on the other side of AB.

(3) A force is given by a length 3·48 inches (scale 10 lbs. to 1·25 cm.), find graphically its moments about points distant 6·72 ft. from it on opposite sides of the force axis.

(4) Find the moments in Ex. 3 by drawing through O lines parallel to PA and PB and measuring the intercept on ab.

Another Graphical Construction. With O as centre—on your drawing for Fig. 263—describe a circle of 1″ radius. From any point Q on ab draw QO and a tangent QT to this circle. Find the components of **AB** in ab along QO and QT. Measure the latter component on the cm. scale; it is the moment of **AB** about O in lbs. ft. units.

Proof. The sum of the moments of the components of a force is equal to the moment of the force itself (see p. 296). As one component passes through O and the other is at unit distance from it, the second component must measure the moment.

Sum of Moments of Like Parallel Forces.

EXAMPLE. *Draw four parallel lines distant apart* 1·22, 2·38 *and* 1·94 *inches, and let downward forces represented by lines of* 3, 4, 2·5 *and 3·7 cms. (scale 1″ to 10 lbs. weight) act in these. Take a point* O *1·43 inches to the left of the first axis. Find the sum of the moments of these forces about* O.

Add the vectors of the forces, and choose a pole P distant 3 inches from the vectors (Fig. 264).

Draw the link polygon $R_1R_2R_3R_4R$ in the usual way and produce the links to cut the line through O parallel to ab in $X_1, X_2, \ldots, X_5$.

Measure X_1X_5 in inches and multiply by 30; the product is the sum of the moments in lbs. inches.

Proof. Let x_1, x_2, x_3 and x_4 be the distances of O from ab, bc, cd, de. Since the $\triangle PAB$ is similar to $R_1X_1X_2$,

$$\therefore \frac{AB}{h} = \frac{X_1X_2}{x_1}, \quad i.e. \; AB \,.\, x_1 = hX_1X_2.$$

Again, PBC, PCD, PDE are similar to $R_2X_2X_3$, $R_3X_3X_4$, $R_4X_4X_5$, respectively;

$$\begin{aligned} \therefore \; BC \,.\, x_2 &= h \,.\, X_2X_3, \\ CD \,.\, x_3 &= h \,.\, X_3X_4, \\ DE \,.\, x_4 &= h \,.\, X_4X_5. \end{aligned}$$

Adding, we get

$$\begin{aligned} \text{Sum of the moments of the forces about } O \\ &= h(X_1X_2 + X_2X_3 + X_3X_4 + X_4X_5) \\ &= hX_1X_5. \end{aligned}$$

And since X_1X_5 is downwards, the sum is negative and the moment clockwise.

Obviously, for more than four forces we only need to extend the construction; the method will be exactly the same.

The sum of the moments is thus represented by a rectangle of height X_1X_5 and base h. The moment of 1 lb. inch is represented by a rectangle of height 0·1″ and base 1″; hence, since $h = 3''$, to reduce $h \,.\, X_1X_5$ to unit base we must treble the altitude, *i.e.* $3 \,.\, X_1X_5$ measured on the tenth inch scale gives the sum of the moments in lb. inches.

The moment of the resultant force **AE** *acting through* R *is equal to the sum of the moments of the components.*

If x is the perpendicular distance from O on the axis of AE,

then $$\frac{X_1X_5}{x}=\frac{AE}{h} \text{ or } AE \,.\, x=h \,.\, X_1X_5.$$

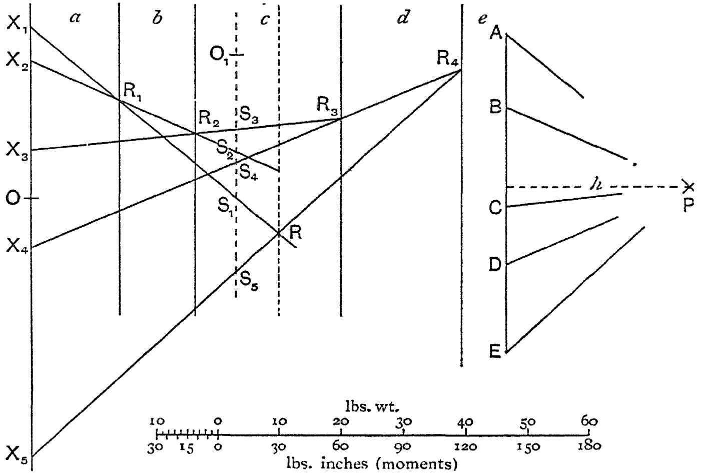
X₁
X₂
X₃
O
X₄
X₅
a
b
c
d
e
O₁
R₁
R₂
S₃
S₂
S₄
S₁
R
S₅
R₃
R₄
A
B
C
D
E
h
P
lbs. wt.
10 0 10 20 30 40 50 60
30 15 0 30 60 90 120 150 180
lbs. inches (moments)

FIG. 264.

If the point about which moments are to be taken is at O_1, in the space c, then the moment of **AB** is $h \,.\, S_1S_2$ and is positive; the moment of **BC** is $h \,.\, S_2S_3$ and is positive; the moment of **CD** is $h \,.\, S_3S_4$ and is negative; and the moment of **DE** is $h \,.\, S_4S_5$ and is negative: the algebraic sum of the moments is thus $h \,.\, S_1S_5$ or the intercept between the first and last lines of the link polygon multiplied by h.

For all positions of O then, the intercept, between the first and last lines of the link polygon multiplied by h, measures the sum of the moments.

A simple inspection of the figures shews that this sum must always be equal to the moment of the resultant.

Fig. 264 has been drawn for parallel forces having the same sense; the conclusion applies to all parallel forces which have a resultant.

The sum of the moments of a number of parallel coplanar forces about any point in their plane is equal to the moment of the resultant about that point. The algebraic sum of the moments is given by the intercept, between the first and last lines of the link polygon, on a line drawn through the point parallel to the forces.

Notice that, *the sum of the moments about any point in the resultant is zero.*

(5) Find the sum of the moments about points in the spaces *a, d* and *e*.

Sum of Moments of Unlike Parallel Forces.

EXAMPLE. *Draw six parallel lines* ab, bc, cd, de, ef *(from left to right), the spaces* b, c, d, e *and* f *being* 0·82, 1·2, 1·46, 1·79 *and* 1·13 *inches wide; forces of* 4·6 *(down),* 1·5 *(up),* 5·45 *(down),* 2·8 *(down),* 5 *(up) and* 3·2 *(down) lbs. weight act in these lines. Find the sum of the moments about a point distant* 1·18 *inches to the left of* ab.

Take a pole P (Fig. 265) at 2 inches distance from the sum **AE** of the vectors, and proceed exactly as before, the only difference being that the link polygon is now re-entrant. Measure X_1X_7 on the force scale and double the number obtained; the result is the sum of the moments in lbs. inches.

(6) Parallel forces of 3·3, 4·1, 2·3, 1·5 and 2·8 tons weight act on lines distant 7, 8, 4 and 6 ft. apart from left to right, the first and last forces being downwards and the rest upwards; find the sum of the moment in ton ft. units about the points,

(i) distant 3 ft. to right of axis on extreme right,

(ii) distant 4·5 ft. from first and 2·5 from second axis,

(iii) distant 1·8 ft. from third and 2·2 ft. from fourth axis.

(7) Find separately in cases (ii) and (iii) the sum of the moments of all the forces on the left and on the right of the point.

(8) The distances apart of the centres of the wheels of an express engine and tender are 9′ 8″, 5′ 3″, 6′ 0″, 11′ 2·5″, 6′ 6″ and 6′ 6″ from the leading wheels backwards. The loads carried by these wheels are 14 tons 10 cwts., 17 tons 8 cwts., 14 tons, 14 tons 10 cwts., 12 tons 5 cwts., 12 tons 10 cwts. and 13 tons 5 cwts. The engine is partly on a bridge, one bridge support being mid-way between the centres of the fourth and fifth wheels. Find the sum of the moments of the loads about the support.

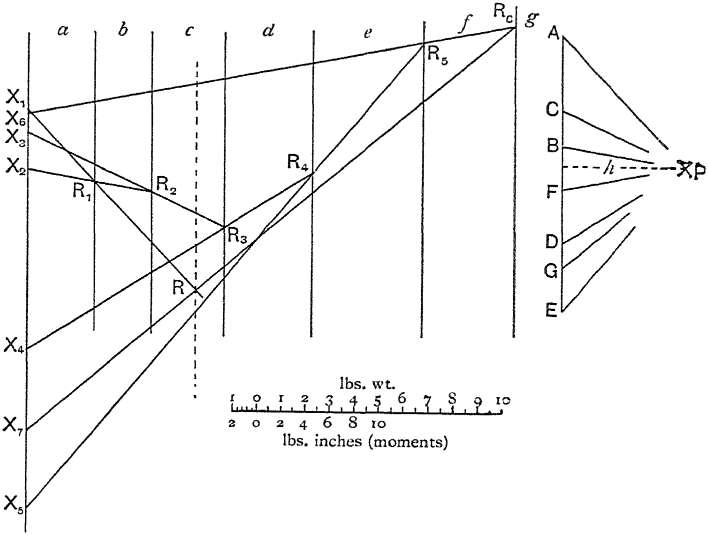

FIG. 265.

Sum of the Moments of Parallel Forces in Equilibrium. The sum of the moments of such a set of forces is zero for all points in their plane, and the sum of the moments of all the forces on one side of a point is equal in magnitude—but opposite in sense—to the sum of the moments of all the forces on the other side.

Any one force of the given set reversed in sense is the resultant of the rest. Its moment, for all points, is minus the moment of the resultant and this is equal to the sum of the moments of the rest. Hence the sum of the moments of the given set of forces is zero. A proof from the link polygon is given on p. 292.

EXAMPLE. *A locomotive has the centres of the wheels from front to rear at the following distances apart,* 8′ 9″, 5′ 5″, 5′ 5″, 6′ 0″. *The loads borne by these wheels are* 6 *tons* 8 *cwts.*, 14 *tons* 6 *cwts.*, 14 *tons* 8 *cwts.*, 16 *tons* 7 *cwts.*, 16 *tons* 7 *cwts.*; *the engine is on a freely supported bridge of length* 40 *ft.*, *and the leading wheel is at a distance of* 9′ *from the left-hand pier. Find the sum of the moments of all the forces to the left of a point mid-way between the third and fourth wheels about that point.*

Draw the reaction and load lines of the bridge to scale (say, 1 cm. to 1 ft.); then set out the load vectors **AB**, **BC**, **CD**, **DE** and **EF** to a scale of (say) 1 inch to 10 tons (Fig. 266). Take a pole P at a distance of 20 cms. from AF, and draw the link polygon $XR_1R_2R_3R_4R_5Y$; X and Y being the points on the reaction lines.

Since there is equilibrium, XY must be the closing line, and the reactions are determined by drawing PO in the vector polygon parallel to XY.

Through Z, the mid-point of the space d, draw a line parallel to the axes and cutting XY in M and R_3R_4 in N.

Measure MN on the force scale and multiply by 20; the product is the sum of the moments, about Z, of all the forces to the right or left (taking account of sense) of Z.

Proof. To find the sum of the moments of all the forces to the left of Z, we find the intercept between the first and last line of the link polygon. The first force (the reaction) has **OA** as its vector, and therefore the first line of the link polygon (drawn according to rule) is XY, the second is X_1R_1, the third R_1R_2, the fourth is R_2R_3 and the final (taking into account only forces to the left of Z) is R_3R_4. Hence MN gives the sum of the moments of all forces to the left of Z, its sense is downwards and the moment therefore negative.

The unit moment of a ton ft. is represented by a rectangle of height 0·1 inch and base 1 cm., the sum of the moments is given, from the construction made, by a rectangle of height MN and base 20 cms. If we measure MN, therefore, on the force scale and multiply by 20, we have the sum of the moments in tons ft.

(20 cm. was taken as h instead of 10 to avoid the lines of the link polygon being too steep.)

For the sum of the moments about Z of all the forces on the right we have R_3R_4 as the first link, and since **FO** is the last force, XY is the last link, and the intercept is now NM. The sum of the moments is, therefore, of the same magnitude, but of the opposite sense.

The total sum of the moments is therefore zero.

Evidently, *the deduction is true for all parallel forces in equilibrium.*

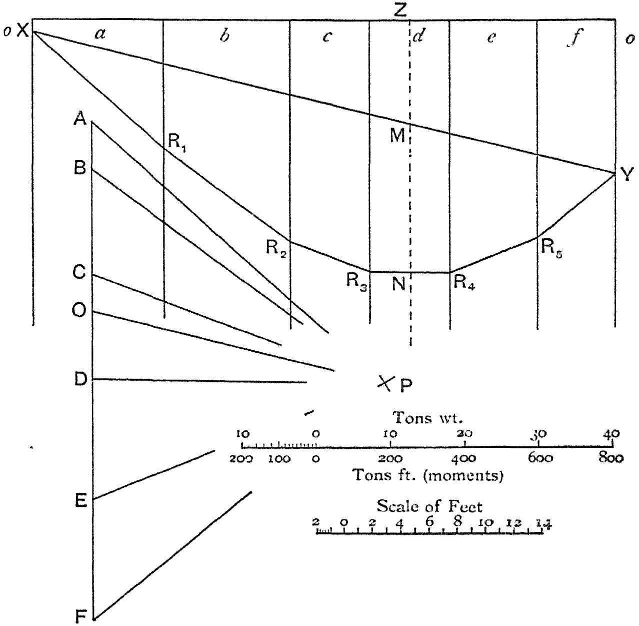
Z
o X
a
b
c
d
e
f
o
A
B
C
O
D
E
F
R₁
R₂
R₃
N
M
R₄
R₅
Y
P
Tons wt.
10 0 10 20 30 40
200 100 0 200 400 600 800
Tons ft. (moments)
Scale of Feet
2 0 2 4 6 8 10 12 14

FIG. 266.

(9) The distances apart of the centres of the wheels of an express engine and tender are (from the leading wheel backwards) 12′ 0″, 10′ 0″, 8′ 7·25″, 6′ 9″ and 6′ 9″. The loads borne by these wheels are *21 tons 15 cwts.,

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* This load of 21 tons 15 cwts. is really that borne by the two pairs of bogie wheels; they are taken as one load to make the example simpler. The first distance, namely 12′ 0″, is from the centre of the bogie truck to the front driving wheel.

19 tons, 19 tons, 12 tons, 12 tons 5 cwts., 12 tons 15 cwts. The engine stands on a bridge, the left-hand support being 11′ 6″ from the leading wheel (centre), and the bridge is 75 ft. long. Find the sum of the moments of all the forces to the left of the centre of the bridge about the centre.

Two Parallel Forces. (i) *Of same sense.* This is only a special case of the general construction, but it is worthy of separate consideration.

Draw (Fig. 267) any two parallel lines *ab* and *bc*, and vectors **AB**, **BC** of the forces supposed to act in those lines. Construct the axis of the resultant in the usual way and mark R_1, R_2, R and R_3 where the axes of the forces cut the links. $R_3R \times h$ then gives the moment of the force in *bc* about any point in axis of resultant, and $RR_3 \times h$ gives the moment of the force in *ab*.

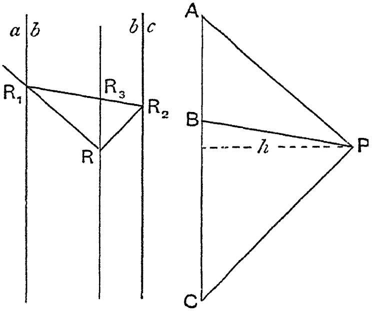

FIG. 267.

From the triangles R_1R_3R and R_3RR_2, which are similar to triangles in the vector diagram, we obtain

$$\frac{R_1R_3}{R_3R}=\frac{BP}{AB} \text{ and } \frac{R_3R_2}{R_3R}=\frac{BP}{BC},$$

and therefore

$$\frac{R_1R_3}{R_3R_2}=\frac{BC}{AB},$$

or, the resultant divides the distance between the axes inversely as the magnitudes of the forces.

Notice that R_1R_2 is any line; if, then, the axes turn round any points R_1 and R_2 and remain parallel, the axis of the resultant turns round a fixed point in R_1R_2, viz. R_3.

(ii) *Of opposite senses.* Construct as in the previous case; now *BC* is upwards and the axis of the resultant is external to the other axis and nearer the greater force.

See from the similar triangles that the axis of the resultant divides externally the distance between the other axes inversely as the magnitude of the forces.

(iii) *Equal in magnitude but opposite in sense.* This is the case already considered in the chapter on the link polygon (p. 184).

(10) The axes of a couple are 3·76″ apart, each force is of magnitude 7·21 lbs. weight. Take the pole at unit distance from the vector line and find a line giving the momental area of the couple in lb. inches.

(11) With the same forces as in Ex. 10 find a line giving the momental area of the couple in lb. centimetres.

(12) If a kilogramme = 2·204 lbs. find the momental area of the couple in Ex. 10 in kil-centimetre and kil-inch units.

Moments of a Couple.—Direct Proof. *The sum of the moments of a couple of forces is the same for all points in the plane and is equal to the momental area of the couple.*

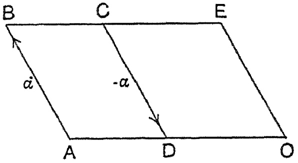

FIG. 268.

a and $-a$ being the forces and O (Fig. 268) any point, the moment of a about O is given by $OABEO$, that of $-a$ about O is given by $OECDO$. The algebraic sum of these is $ABCD$, which is the momental area of the couple, and this result is quite independent of O.

The moment and the couple are two distinct things. The couple is simply the pair of forces, the momental area of the couple measures the sum of the moments of the pair of forces about every point in the plane.

Sum of Moments for Concurrent Forces. Draw any parallelogram $OACB$ (Fig. 269) and let OA and OB represent concurrent forces, then OC represents their resultant. Take points P, P_1 and P_2 as indicated, and measure the perpendiculars from them on OA, OB and OC. Find the algebraic sum of the moments of OA and OB about P, P_1 and P_2, and compare with the moments of the resultant OC about those points.

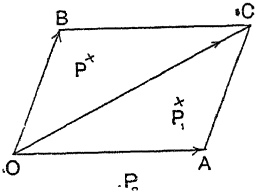

FIG. 269.

(13) Draw any three non-concurrent lines and take an arbitrary vector polygon for the forces in them. Find the axis of the resultant by the link polygon. Measure the forces and resultant to any scale. Mark some

point on the paper and measure the perpendiculars from the point on the four forces and calculate the moments. What is the connection between these moments?

The algebraic sum of the moments of two concurrent forces is equal to the moment of the resultant about all points in their plane.

Proof. Let *a*, *b* and *c* (Fig. 270) be the axes of the two forces and their resultant, the senses being as indicated. *P* is any point in their plane.

Through *P* draw a line parallel to *c*, cutting *a* and *b* in *A* and *B*.

Then *AB* may be taken to represent the resultant in magnitude, direction and sense, and *AOB* is the vector polygon for the forces.

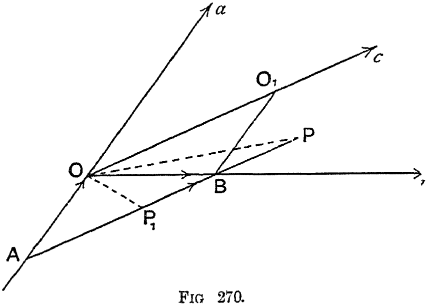

FIG 270.

Then the moment of **AO** about *P* is twice *AOP* and is positive, the moment of **OB** about *P* is twice *OBP* and is negative; therefore their algebraic sum = twice *AOB* and is positive.

(Notice that the sum is the same for all points in *AB* or *AB* produced, and that, whatever the position of *P*, it is always twice the area of *AOB* and in the sense of the letters.)

Draw BO_1 parallel to *AO*; then twice $AOB = AOO_1B$ and therefore measures the moment of **AB** in *c* about *A*, *B* or any point *P* in *AB*.

It will be seen that the proof is perfectly general for all positions of *P*.

For any system of coplanar forces the sum of the moments of the components is, for all poles, equal to the moment of the resultant (if there is one), or to the momental area of the resultant couple (if there is one) or is zero.

Proof. This result follows at once from the link polygon construction, which consists in finding the resultant of concurrent forces two at a time, and since the theory of moments holds at each new composition it must hold at the final step when the resultant or resultant couple is found.

As a particular case consider the decomposition on p. 203. Taking moments about X, we have moment of **AB** = moment of **AD**, and hence **AD** is uniquely determined.

Sum of the Moments of any Number of Forces about a Point. If the forces have a resultant, take the pole of the vector polygon at unit distance (or a simple multiple thereof) from the resultant vector. Measure, on the force scale, the intercept between the first and last lines of the link polygon on a line drawn through the given point parallel to the resultant vector.

If the forces are equivalent to a couple (the vector polygon closed) take a force of unit magnitude (or a simple multiple thereof) as the arbitrary force vector and measure, on the distance scale, the perpendicular between the first and last lines of the link polygon.

(14) The magnitudes of five forces are given by 3·7, 4·8, 6·1, 2·8 and 5·3 cms., the scale being 1·7 inches to 10 lbs. weight. The coordinates of points on their axes are (0, 0), (1, 2), (3, 2), (1, 4), (2, 3) inches respectively, and the forces are directed towards N. (the y axis), N.E., 15° N. of E., S.W. and W. Find the sum of their moments in lbs. inches about a point whose coordinates are (1·5, 2·7), the coordinates being all measured in inches.

(15) Choose the magnitudes of the first and last forces in Ex. 14, so that the vector polygon is closed. Find the momental area of the equivalent couple.

(16) Find the sum of the moments of three forces of magnitudes 3, 7·2 and 5 lbs. weight acting along AB, BC, CA the sides of a triangle about a point whose coordinates are (4, 2) inches. The coordinates of the vertices A, B and C of the triangle are (2, 2), (1, 3), (3, 4) inches.

(17) Draw a triangle ABC of sides $AB=3$, $BC=3{\cdot}5$ and $CA=4$ inches; through the vertices draw three parallel lines, and suppose forces of magnitudes 6·2, 7·4 and 9 lbs. to act in these lines through A, B and C and to have the same sense. Construct the line of action of the resultant in the usual way. Draw three more parallel lines through A, B, C, making 45° with the first set, and suppose forces of the same magnitude as before

to act in them; construct the new axis of the resultant. Repeat the construction for parallel lines which are perpendicular to the first set.

See that the three axes of the resultants are concurrent. This point of concurrence is called the **centre of the parallel forces.**

(18) The coordinates of four points are (1, 2), (0, 3), (4, 0) and (2·5, 3·6). Parallel forces of like sense act through these points and are of magnitudes 2·2, 3·5, 1·8, 3 lbs. weight. Construct, as in previous example, three link polygons, the sides making 45° with each other respectively, and shew that the three axes of the respective resultant forces are concurrent.

(19) Three parallel forces equal in magnitude and of same sense act through the vertices of a triangle. Shew by three constructions that the resultant passes through the M.C. of the triangle.

Centre of Parallel Forces.

A number of parallel forces pass through points A, B, C, ... respectively. If the axes turn about these points so as all to remain parallel, then the axis of the resultant turns about a fixed point in itself, the centre of the parallel forces.

Suppose masses to be at A, B, C, ..., whose magnitudes have the same numerical values as the forces acting through the points. The mass-centre of these mass-points must lie on the axis of the resultant force, for the sum of the moments of the forces and of the masses have exactly the same numerical value. Supposing, then, the axes of the forces to have a different direction, the M.C. of the mass-points must be at the intersection of the axes of the resultants. But the masses can have only one M.C., and, therefore, whatever direction the parallel forces may have, the resultant must always pass through a fixed point, viz. the mass-centre of the masses.

Since the mass-centre theorem remains true if some of the points be considered as having negative masses, the above theorem remains true if some of the forces have a different sense from the rest.

(20) Parallel forces of like sense and of magnitudes 1·27, 2·18, 3·24, 4·1 lbs. weight act through the corners $ABCD$ of a square of side 8·35 cms.; find the centre of the parallel forces.

(21) Find the centre of the parallel forces in the last exercise if the 3·24 lbs. weight force be reversed in sense.

(22) Draw a triangle ABC having $AB=10·2$, $BC=11·8$, $CA=6·48$ cms. Parallel forces act through A, B and C of magnitudes (in lbs. weight) given by the opposite sides. Find graphically the centre of the parallel forces and shew that it is the centre of the circle inscribed in ABC.

Mass-Centres by Link Polygons. The mass-centres of a number of mass-points can be accurately and expeditiously found by the aid of the link polygon construction.

EXAMPLE. *Masses of* 1·32, 1·66, 2·15 *and* 1·67 *lbs. are concentrated at points whose coordinates in inches are* (1·14, 0), (2·2, 1·13), (3·36, 2·87), *and* (4, 2), *respectively. Find the position of the mass centre.*

Plot the points on squared paper (Fig. 271), and through them draw lines parallel to the axes of coordinates; label those parallel to the y axis m_1, m_2, m_3, m_4 in order from left to right, then those parallel to the x axis from top to bottom must be labelled m_4, m_2, m_1, m_3.

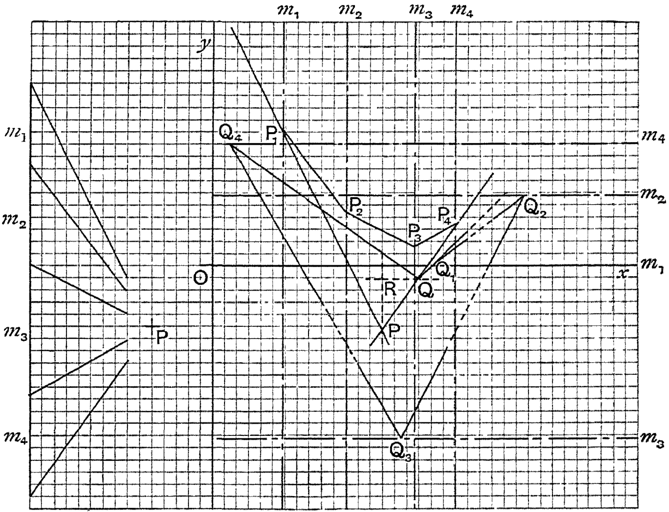

FIG. 271.

Set off the masses m_1 $(=1·32)$, m_2 $(=1·66)$, m_3 $(=2·15)$, m_4 $(=1·67)$ to scale along a line parallel to the y axis, and choose some convenient pole P' for the vector polygon.

Using one side of a set square bounding a right angle, draw through P_1, any point in the **vertical** m_1 line, a link PP_1 parallel to the first line of the vector polygon. With the other edge of the set square draw through any point Q, in the **horizontal** line m_1, a link QQ_1 perpendicular to the first line of the vector polygon.

In this way it is quite easy to draw correctly and quickly two link polygons $P_1P_2P_3P_4$, $Q_1Q_2Q_3Q_4$ whose vertices lie on the vertical and horizontal m lines, and whose corresponding links are perpendicular.

The intersection at P of the first and last links of the polygon $P_1P_2P_3P_4$ gives a line PR, parallel to Oy, on which the M.C. of the points must lie, and the intersection at Q of the first and last links of the polygon $Q_1Q_2Q_3Q_4$ gives a horizontal line QR, on which the M.C. must also lie. The point R of the intersection of these lines is therefore the M.C.

Proof. Since the M.C. of a number of masses is the same as the centre of the parallel forces whose magnitudes are proportional to the magnitude of the masses, all we have to do to find the former point is to find the axis of the resultant force in two cases. This is done most conveniently by supposing the forces acting through the mass-points to be (i) parallel to the axis of y, (ii) parallel to the axis of x.

To find the axis in the first case the vectors of the weights are drawn parallel to the y axis, and a link polygon constructed, and the resultant force has PR as its axis.

To find the axis in the second case, the vectors of the loads may be drawn parallel to the x axis and a second link polygon constructed. It is, however, more convenient, instead of drawing a fresh vector polygon, to suppose the first one turned through a right angle. To construct the second link polygon, therefore, we have only to draw links perpendicular to those of the first link polygon. There is less chance of error if the two link polygons be constructed simultaneously, by aid of two perpendicular edges of a set square, than if one be drawn completely first.

To find the M.C. it will, in general, be necessary to draw two link polygons, preferably at right angles, each determining a line on which the M.C. must lie.

(23) Masses 3, 2·5, 5, 4, 3·7 lbs. are concentrated at points whose coordinates are (1, 0), (2, 3), (1, 1), (3, 2) inches. Draw two link polygons at right angles and find the mass-centre.

Test the accuracy of your results by taking moments about the axes of coordinates.

(24) Choose another pole for the vector polygon in the previous question and see that the same point is obtained for M.C.

(25) Masses given by lines of length 2·7, 1·86, 3·1, 1·72, 0·94 inches are concentrated at the vertices, taken in order, of a regular pentagon of side 3 inches. Find graphically the position of the M.C. and test roughly by measurement and by calculating the moments.

(26) Draw as follows five straight line segments to form a broken or zig-zag line: (i) horizontally a length of 4·8 cms., (ii) sloping upwards at 45° a length of 3·75 cms., (iii) sloping downwards at 15° a length of 5·3 cms., (iv) sloping downwards at 60° a length of 2·8 cms., (v) horizontally a length of 4·25 cms. Find graphically the position of the M.C. of the zig-zag line. (Suppose each line to be concentrated at its mid-point.)

(27) Find graphically the M.C. of six sides of a regular heptagon. (Notice there is an axis of symmetry in which the M.C. must lie.)

M.C. of Areas by the Link Polygon. When the area can be divided up into parts for which the M.C.'s are found easily we may apply the link polygon to find the M.C. of these mass-points.

At each of these M.C.'s we must suppose a mass concentrated proportional to the corresponding area.

EXAMPLE. *Find the* M.C. *of the area given by Fig.* 272.

Draw the figure to scale and mark its axis of symmetry.

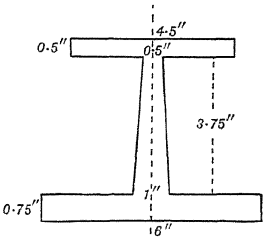

FIG. 272.

Divide the area up into top and bottom rectangles and construct their M.C.'s.

Construct the M.C. of the central trapezium.

Draw horizontal lines through the M.C.'s of the two rectangles and the trapezium. Reduce the areas of the three parts to unit base and draw a vector polygon for masses proportional to the areas, and finally, by a link polygon, determine the horizontal line on which the M.C. of the whole must lie.

(28) Find the M.C. of the double angled iron in Fig. 273. (Divide up into four rectangles and use two link polygons.)

(29) Find the mass-centre of the bar shewn in Fig. 274; the ends are semi-circles of radii 2·9 and 2·1 cms., the distance of the centres apart being 12·1 cm.

(30) Find the M.C. of the area in Ex. 29 when a circular hole of radius 1 cm. is cut out as indicated by dotted circle.

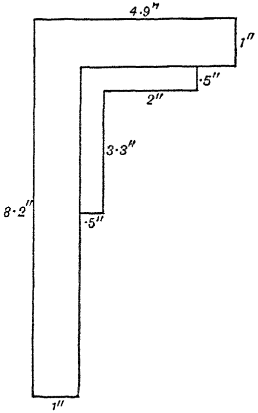

FIG. 273.

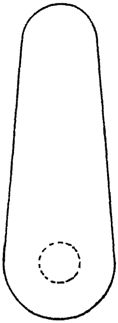

FIG. 274.

Irregular Areas. When the area is irregular, or cannot be divided into parts for which the M.C.'s are known, we may resort to the method of strip division. Divide the area up into a number of equally narrow strips, take the mid-point of the middle line of each of these strips as the M.C. of the strip, and draw two link polygons for these mass-points. The masses at the points are approximately proportional to the lengths of the mid-lines of the strips. If the area has an axis of symmetry only one link polygon need be constructed.

(31) Draw a semi-circle of radius 4″, divide up into ten equally wide strips parallel to the base, and determine the M.C. of these strips by the link polygon method.

(32) Find the M.C. of the irregular figure given on p. 58.

Centre of Gravity, Centre of Parallel Forces, Mass-Centre, Centroid, Centre of Figure, Centre of Mean Position. Every particle of a body near the earth's surface is attracted towards the centre of the earth. The body being small compared with the earth, the axes of these forces are parallel (or nearly so). The centre of these parallel forces is called the **centre of gravity** of the body. The centre of gravity as thus defined is the same as the **mass-centre** of the body. Moreover, if the mass be uniformly distributed throughout the volume, it is the same as the **centroid** or **centre of figure**, and as the **centre of mean position**.

Centre of gravity is not, however, a good term to use, since it denotes a point whose position depends not only on the body but on the earth also, whereas the mass-centre depends on the body alone and would remain unaltered if the body could be taken right away from all external forces.

Since the mass-centre or centre of gravity of a body is the point through which the resultant of the weights of the particles always acts no matter what the position of the body (near the earth's surface), the body, if supported at that point, would be in equilibrium.

This consideration leads to an easy experimental way of finding the M.C. of many bodies. Suppose a triangular board *ABC* (Fig. 275) suspended by a string attached to any point *D* of it; then, since only two forces act on the board, viz., the pull of the string and the resultant weight, these must be in a line, and therefore if a line is drawn on the board in continuation of the string it will pass through the M.C. By suspending the board from another point D_1 and marking the point where the vertical through D_1 cuts the line already on the board, the position of the M.C. is determined.

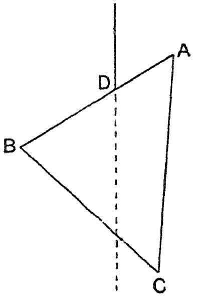

FIG. 275.

The M.C. of such bodies as cardboard or wooden triangles, quadrilaterals, circular sectors, ... should be determined experimentally, and the results compared with the graphical determinations.

***Moment and Couple.** The moment of a force a in AB about a point O is the same as the momental area of the couple a in AB and $-a$ in CO (where CO is parallel to AB), since both are given in magnitude and sense by the parallelogram $OABC$.

***Resultant Couple and Moments.** Any set of forces (coplanar) can be reduced to a resultant force through any chosen point O and a number of couples whose momental areas are added algebraically to the momental area of the resultant couple. This couple has therefore a momental area given by the sum of the moments of the forces about O.

***Moments and Equilibrium.** If there is equilibrium, the resultant force and the resultant couple must vanish for any point O, hence the sum of the moments about any point must be zero.

***Moments of Resultant and Components.** If a system of forces has a resultant, this resultant reversed in sense must be in equilibrium with the components, and therefore the sum of the moments of the given forces minus the moment of the resultant must be zero for every point. Hence,

Σ moments of components = moment of resultant for all points.

***Theory of three Moments.** Any system of coplanar forces can be reduced to either

(i) a resultant force,
(ii) a resultant couple,
(iii) or there is equilibrium.

If, then, the sum of the moments of the forces about one point be zero, there is either equilibrium, or there is a resultant force passing through the point. (There cannot be a resultant couple because the sum of the moments = the momental area of the couple.)

If, then, the sum of the moments is zero for three non-collinear points, the forces are either in equilibrium, or there is a resultant passing through three non-collinear points. The latter alternative being impossible the forces must be in equilibrium.

MISCELLANEOUS EXAMPLES. VIII.

1. *ABCD* is a rectangle, $AB=12$, $BC=8$. At *A*, *B*, *C* and *D* are masses 8, 10, 6 and 11 lbs. Find the M.C. by the funicular polygon. (B.Sc., 1905.)

2. Prove that the sum of the moments of two forces in a plane about any point in their plane is equal to the moment of their resultant about that point. Can the conditions of equilibrium of a body acted on by a system of forces in one plane be expressed solely by the principle of moments? (Inter. Sci., 1906.)

3. What do you understand by "the moment of a force"? A downward push of 40 lbs. acts on a 6″ bicycle crank which is 50° below the horizontal position. What is the magnitude of the moment produced about the axis? If, by suitable ankle action, a push is produced at right angles to the crank in the same position, how great must this be to produce the same moment as the downward push of 40 lbs.? (Naval Cadets, 1904.)

4. Indicate the method of finding the resultant of two parallel, unequal, unlike forces acting upon a rigid body.

A uniform bar 12 feet long, weighing 56 lbs., rests horizontally upon two supports, one being under one end *A* and the other being 5 feet from the other end *B*; supposing a weight of 10 lbs. to be hung from the end *B*, find the pressures on the two supports. (B. of E., Stage I., 1904.)

5. Define the moment of a force about a point and state any theorem concerning moments.

ABC is a triangle with a right angle at *A*, *AB* is 2 feet and *AC* is 3 feet; a force of 5 lbs. acts from *A* to *B* and one of 4 lbs. from *A* to *C*; find the moment of the forces about the middle point of *BC*.

If the point in question were fixed, indicate on a diagram the direction in which the triangle (supposed to be a lamina) would revolve.

6. When a body capable of turning about an axis is at rest under the action of two forces perpendicular to the axis, what is the relation between these forces? (State the relation, no proof is wanted.)

The disc in Fig. 276 weighs 2 lbs. and turns about the point *O*. What force *P*, acting in the position shown, is required to hold the disc in the position shown? (Naval Cadets, 1903.)

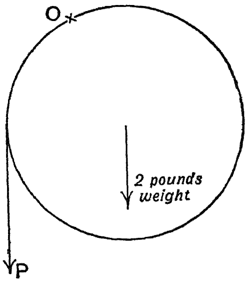

FIG. 276.

7. Fig. 277 represents a form of wheel and axle, drawn to the scale of one-tenth. A man sits on a platform suspended from the end *A* and raises himself by pulling on the end *B*. What force must he exert to support himself, and how much work must he do to raise himself a distance of 2 metres? The weight of the man with the platform is 100 kilogrammes, and the pull at either end of the tackle to raise a given weight at the other is twice as much as it would be without friction. (Military Entrance, 1905.)

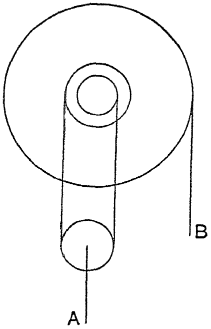

FIG. 277.

8. An hexagonal table, diameter 3 ft. and weighing 50 lbs., has weights 5, 10, 15, 20, 25 lbs. placed in order, etc. at five of the angles. Determine the centre of the parallel forces, and its distance from the centre of the table. (B. of E., II., 1904.)

9. Mark five points in a line *PQRST*, the distances apart representing 3·2, 4·7, 1·8, and 2·6 ft. from left to right. Through *P*, *Q*, *R*, *S* and *T* act forces of magnitude 1510, 2150, 750, 1830 and 1980 lbs. weight. The forces make angles of 15°, 50°, 80°, 140° and 250° with *PT* reckoned contra-clockwise.

Find the sum of their moments in lbs. weight about a point *U*, distant 7 ft. to the left of *P*, (i) by decomposing each force into two components, one of which is parallel to *PT*, and the other passes through the point *U*; (ii) by finding the resultant force.

10. On squared paper mark five points whose coordinates are (2·1, 3·3), (0·3, 5·2), (3·7, 2·1), (3·9, 4·5), (5·7, 0·8) inches. Masses given by lines of length 2·35, 1·82, 4·16, 3·05, 2·18 centimetres (scale 15 cms. to 10 lbs.) are at these points. Find the coordinates of the mass-centre by construction.

CHAPTER IX.

BENDING MOMENT AND SHEARING FORCE.

Hooke's Law. If a bar be subjected to tensile or compressive stress its length changes; the relation between the stress and the elongation, or compression, was discovered by Hooke, and is usually called Hooke's Law. The law is purely an experimental one.

If l be the original length of the bar, T the force producing extension, and e the elongation, then T is proportional to $\frac{e}{l}$, or

$$T = \lambda \frac{e}{l}, \quad \text{(Hooke's Law)}$$

and λ is called the **modulus** of the bar.

If A is the cross sectional area of the bar, then

$$\frac{T}{A} \text{ is the stress per unit area,}$$

and
$$\frac{T}{A} = \frac{\lambda}{A} \cdot \frac{e}{l}.$$

If $\frac{\lambda}{A} = E$ then E is called **Young's Modulus** for the material of which the bar is made.

An exactly similar law holds for compression. Always, then, if a bar is in a state of compressive or tensile stress it is shorter or longer than its natural length, and this stress is proportional to the compression or extension.

For very large forces the law ceases to hold and the **elastic limit** of the material is said to have been passed.

This law is of great importance in many ways; it has important bearings on the stresses set up in many frames as well as on the bending of beams.

Simple Cantilever. *AB* (Fig. 278) represents a horizontal beam fixed in a wall at *A* and loaded at its free end *B* with a weight *W*. (The weight is supposed so large that the weight of the beam itself may be neglected in comparison with it.) Consider the equilibrium of the part *BP* of the beam. It is evident that the force or forces which the part *AP* exerts on *PB* must be in equilibrium with the load *W* at *B*. Suppose the beam cut vertically through at *P*, then the forces which we have to apply to the cut surface at *P* to keep *PB* in equilibrium must be equivalent to the reactions of *AP* on *PB*.

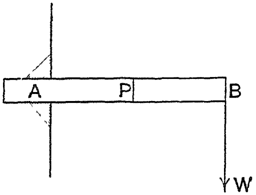

Fig. 278.

No single force at *P* can be equivalent to *W* at *B*; if we suppose a force $-W$ to act at *P*, then *PB* is under the action of a couple whose momental area is $-W.PB$ (clockwise). Hence for the equilibrium of *PB*, we must apply at *P* an upward force of magnitude *W* and a couple of momental area $W.PB$. The reaction forces of *AP* on *PB* must therefore be equivalent to $-W$ at *P* and a couple whose momental area is $W.PB$.

This is simply another way of looking at the theorem on p. 206, viz. a force *W* at *B* is equivalent to a force *W* at *P* and a couple of transference $-W.PB$; it is this force and couple which are equivalent to the reaction forces of *PB* on *AP*.

This theory can actually be demonstrated by connecting the two parts of the beam by a rod *EF*, hinged at its ends, (see

Fig. 279) and a horizontal string DC passing over a pulley G and bearing a load Q, whilst a vertical string at D passes over another pulley and bears a load R.

When the vertical and horizontal pulls on D are adjusted so that BD is horizontal and in a line with C, it is found that the vertical pull R is of magnitude W, and the horizontal pull Q is such that

$$Q \times DF = W \times PB.$$

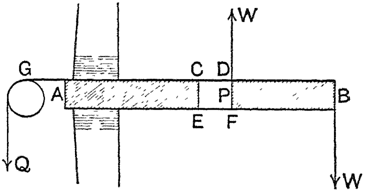

FIG. 279.

Since the part PB, under the action of R, W, Q and the force in EF, is in equilibrium, and since R and W constitute a couple, Q and the force in EF must also form a couple, and therefore EF is in compression and the stress in it is measured by Q.

The upper fibres of the beam (Fig. 278) must therefore be in tension and the lower ones in compression, and hence the upper fibres are elongated and the lower ones shortened. The beam itself must therefore be bent more or less, the loaded end being lower than the fixed one. It is these tensile and compressive forces in the fibres of the beam itself that prevent further bending, and it is the moment of W about P which tends to produce the bending. Hence $W \times PB$ is called the **Bending Moment** at P.

The upward force W at P, of AP on PB, prevents PB sliding downwards relatively to AP, whilst the external load W tends to make it do so, hence W is called the **Shearing Force** at P.

For the portion AP on the left the shearing force and bending moment at P have the same magnitude but are of opposite senses to those on the right.

Bending Moment and Shearing Force Diagram for a Simple Cantilever.

EXAMPLE. *A horizontal beam is fixed in a wall, the length from the wall* A *to the loaded end* B *is* 19·4 *ft. If the load be* 5·18 *tons draw diagrams giving the bending moment and shearing force at every point of the beam.*

Draw *AB* (Fig. 280), of length 3·88″, to represent the beam (scale 1″ to 5 ft.), and then **PQ** the load vector, of length 5·18 cms. (scale 1 cm. to a ton).

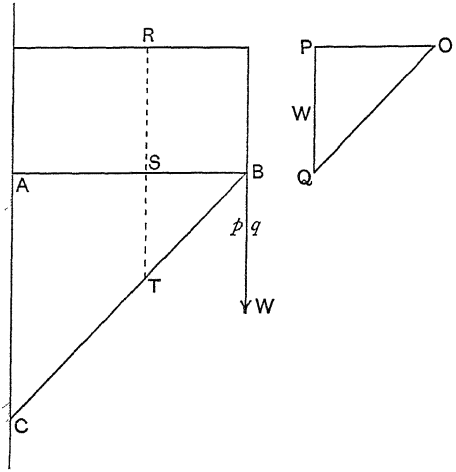

FIG. 280.

Through *P* draw *PO*, of length 2″, perpendicular to *PQ*. Through *B* draw *BC* parallel to *OQ*, and above *AB* draw a rectangle of height *PQ* (*RS* in Fig. 280) and base *AB*.

The triangle *ABC* is the bending moment diagram and the rectangle *RS* is the shearing force diagram.

The bending moment at any point *S* of the beam is given in tons ft. by *ST* measured on the mm. scale.

Draw the force and moment scales and measure the bending moments at points 5·36 and 10·28 ft. from A.

Proof. Consider any point S of the beam; the moment of the load W at B about S is given by (Chap. VIII., p. 288) $ST \,.\, PO$ where ST is parallel to AC.

Since PO represents 10 ft., if ST be measured on the force scale, *i.e.* in centimetres, and multiplied by 10 the result will be the bending moment at S in tons ft.

Hence, wherever S may be in AB, the vertical intercept ST of ABC gives the bending moment (B.M.) in tons ft.

Again, since the load to the right of S is always PQ, the shearing force (S.F.) is constant and, therefore, the diagram for all points is the rectangle RS.

At S the part on the left tends to slide upwards relatively to the part on the right and we may regard SR as being drawn upwards to indicate this. If we wished to indicate that the part on the right tends to slide downwards relatively to the left part then SR would have been drawn downwards.

Authorities differ in this matter. In Cotterill's *Applied Mechanics* the S.F. ordinate, at any point, is set upwards when the part on the left tends to slide upwards relatively to the right hand part and is set downwards in the other case. In the article "Bridges" in the *Encyclopædia Britannica*, on the other hand, the ordinate is set upwards or downwards, at any point, according as the right hand portion tends to move upwards or downwards relatively to the left.

Fig. 281 is an example of Cotterill's method of construction, Fig. 282 an example of the *Encyclopædia* method. With the exception of Fig. 282 we shall adhere to the former way, *i.e.* to Cotterill's method.

B.M. and S.F. diagrams for a beam freely supported at the ends and loaded at any one point.

EXAMPLE. *A beam* LM (Fig. 281), *of length* 21·5 *ft., is supported freely at its ends in a horizontal position. It is loaded with* *a weight* W (3470 *lbs.*) *at a point distant* 13·2 *from* L. *Draw the bending moment and shearing force diagrams.*

Draw *LM* to represent to scale the beam length, and mark the point *N* on it where the load acts. Choose a pole *P* for the vector polygon at 10 units of length from the load vector **AB**.

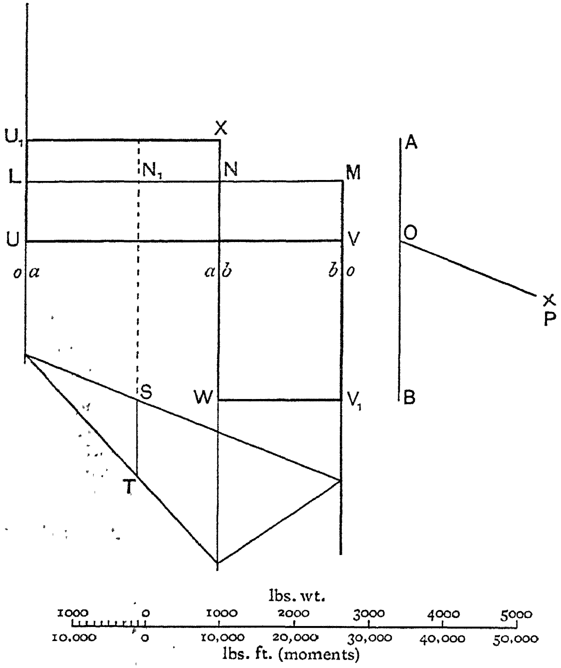
U₁
X
A
N₁
N
L
M
U
V
O
o
a
a
b
b
o
X
P
S
W
V₁
B
T
lbs. wt.
1000
0
1000
2000
3000
4000
5000
10,000
0
10,000
20,000
30,000
40,000
50,000
lbs. ft. (moments)

FIG. 281.

Draw the link polygon and close it; in the vector polygon draw *PO* parallel to the closing line, so that **OA** is the reaction at *L*, and **BO** that at *M*.

At N_1 a point on *LM* where $LN_1 = 7·72$ ft., draw a vertical cutting the link polygon in *S* and *T*.

Measure ST on the force scale and multiply by 10, this product is the bending moment at N_1 in lbs. ft. Draw a scale of bending moments and measure the moment at a point distant 15·7 ft. from L.

Draw a horizontal line UV between the reaction lines at L and M, V being vertically below M. Set downwards $VV_1 = OB$, and upwards $UU_1 = OA$.

Complete the rectangle VV_1WXU_1U as indicated; this is the shearing force diagram. The shearing force at any section is the ordinate of this diagram reckoned from UV.

Proof. Suppose the beam cut through at N_1, then the external vertical force acting on LN_1 is **OA**, and on N_1M it is **AO**, hence the part LN_1 tends to slide upwards relatively to N_1M.

To keep N_1M in equilibrium, we must replace **OA** at L by **OA** at N_1 (the shearing force at N_1) and a couple whose momental area is $OA \,.\, LN_1$. Since this momental area is measured by the moment of OA about N_1 it is given by ST and $10 \,.\, ST$ measures the moment.

Similarly, to keep LN_1 in equilibrium, we must replace **AB** at N and **BO** at M by **AB**+**BO** (=**AO**) at N_1, and a couple of momental area $AB \,.\, N_1N + BO \,.\, N_1B$, *i.e.* by a couple whose momental area is the sum of the moments of **AB** and **BO** about N_1, and this moment is measured by $10 \,.\, ST$.

Hence, before cutting, the material of the beam at N_1 must exert shearing stress given by AO or OA, and stress couples whose momental area is measured by $10 \,.\, ST$.

The shearing stress prevents the shearing of the beam at N_1, LN_1 upwards N_1M downwards; the stress couples prevent the part N_1M rotating contraclockwise, *i.e.* prevent N_1 sagging. Hence the measure of the momental areas of the stress couples at N_1—which prevent the beam bending—is $10 \,.\, ST$.

Similarly, the shear stress at N_1 is measured by OA.

The maximum bending moment is at N, where the load is, the shearing force is constant from L to N, changes suddenly at N and is constant again from N to M.

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However complicated the loading on a beam or girder, the process for finding the shearing force and bending moment at any section is similar to the above.

DEFINITION. **The shearing force (S.F.) at any section is defined as the sum of all the external forces perpendicular to the beam on one side of section, and is considered positive when the right-hand part tends to move upwards relatively to the left-hand part.**

DEFINITION. **The bending moment (B.M.) is defined as the sum of the moments of all the external forces perpendicular to the beam on one side of the section, and is considered positive when the right-hand part tends to rotate or bend contra-clockwise.**

B.M. and S.F. Diagrams when there is more than one Load.

EXAMPLE. *A bridge* 80 *ft. long is supported freely at its ends. The leading pair of wheels (centre line) of a locomotive and tender is* 15 *ft. from one support (abutment) of the bridge, the distances apart of the centre lines of the wheels are* 10′ 5″, 8′ 9″, 10′ 8″, 6′ 6″ and 6′ 6″, *reckoned from the leading wheels. The loads borne by the wheels are* 16 *tons*, 17 *tons*, 16 *tons*, 10 *tons* 7 *cwts.*, 9 *tons*, 9 *tons. The engine and tender being wholly on the bridge, draw the* B.M. *and* S.F. *diagrams.*

Set off the load vectors **AB**, (Fig. 282), **BC**, **CD**, **DE**, **EF**, **FG** to scale. Choose a convenient pole P and draw the link polygon as usual. Close the link polygon and draw PO in the vector polygon parallel to the closing line. The intercept on any vertical line between the first and last lines of the link polygon gives the sum of the moments of all the external forces, including the reaction, on either side of the vertical line.

Hence the link polygon gives the B.M. at any point of the bridge. In this connection the closed **link polygon** is called the **bending moment diagram.**

In Fig. 282 the pole P is taken as four units of length from AG, and hence the B.M. diagram must be measured on the force scale and the measurement multiplied by 4; this gives the B.M. in ton feet.

Draw a horizontal line XY for the datum line of the shearing force diagram. At Y set YZ upwards (see p. 311), equal to GO; at Y_1 on fg set upwards $Y_1Z_1 = FO$; at Y_2 on ef set upwards $Y_2Z_2 = EO$; at Y_3 on de set upwards $Y_3Z_3 = DO$; at X_3 on cd set downwards $X_3W_3 = OC$; at X_2 on bc set downwards $X_2W_2 = BO$; and at X_1 on ab set downwards $X_1W_1 = AO$.

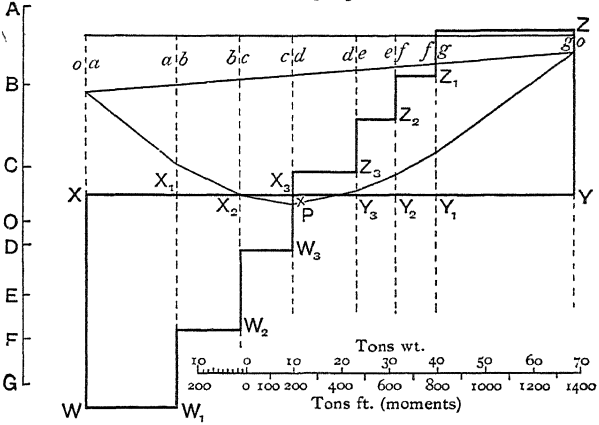

A
B
C
O
D
E
F
G
o a
a b
b c
c d
d e
e f
f g
g o
Z
Z₁
Z₂
Z₃
X
X₁
X₂
X₃
P
Y
Y₁
Y₂
Y₃
W
W₁
W₂
W₃
Tons wt.
10 0 10 20 30 40 50 60 70
200 0 100 200 400 600 800 1000 1200 1400
Tons ft. (moments)

FIG. 282.

Complete the zig-zag $ZZ_1Z_2Z_3X_3W_3W_2W_1W$; it is the shearing force diagram.

Evidently, for the space g, the shearing force is $GO = YZ$, for the space f the sum of the forces to right is **FG + GO**, and the shearing force is FO, and so on all along the bridge.

It will be noticed that the maximum bending moment is along cd, and the maximum shearing force is through the space a.

(1) A horizontal beam fixed in a wall projects 12 ft., and it is loaded at its far end with 500 lbs. Draw the B.M. and S.F. diagrams and measure the B.M. and S.F. at a point distant 4 ft. from the wall.

(2) A cantilever, whose horizontal distance between the free end and the point of support is 25 ft., is loaded at distances of 5, 10, 15 and 25 ft. from its fixed end with 500, 300, 700 and 1000 lbs. weights. Draw the diagrams of B.M. and S.F., and measure these quantities at distances of 8, 15 and 20 ft. from the fixed end.

(3) A beam of length 30 ft. is supported freely at its ends in a horizontal position. Loads of 1, 2·5, 3, 2 tons weight are applied at distances of 6, 10, 20, and 25 ft. from the left-hand end; the beam is propped at the centre, the upward thrust there being equal to a force of 1·8 tons.

Draw the B.M. and S.F. diagrams and measure the B.M. and S.F. at distances of 8 and 20 ft. from the left-hand end.

Bending Moment for non-parallel Forces. In such cases the forces must be resolved into components along and perpendicular to the beam. The former tend to slide the beam off the supports, consequently the beam must be fixed at one end (say by a pin-joint) and supported at the other. The components perpendicular to the beam are alone considered as producing bending moment.

It is not necessary, before attempting to draw the B.M. diagram, to find the reactions at the supports; the B.M. diagram itself determines the components perpendicular to the beam.

(4) A horizontal beam *PQ*, of length 25 ft., is pin-jointed to a support at *P*, and rests freely on the support at *Q*. Forces of 2500, 2000, 1500 and 3000 lbs. weight act at points distant 5, 13, 18 and 20 ft. from *P*, and are inclined to the vertical at angles of 15°, 30°, 60° and 45° towards *Q*. Draw the B.M. and S.F. diagrams, and measure their amounts at points distant 7 and 19 ft. from *Q*.

(5) A beam *PQ*, of length 18 ft., is pin-jointed to a wall at *P* and supported at *Q* by a chain of length 27 ft. which is fastened to the wall at a point *R*, vertically *P*, and distant 12 ft. from it. Loads of 1, 1·2, 2·3 and 1·8 tons are hung at equal intervals along *PQ*. Draw the B.M. and S.F. diagrams.

Reactions non-terminal. Occasionally it happens that the order in which we have to draw the vectors in the vector polygon, to determine the reactions of the supports on the beam, is different from the order of the points on the beam at which the forces are applied. In such problems it is necessary to take care that the links of the link polygon, or B.M. diagram, are drawn between the proper lines. If the diagram is too complicated to be read easily the vector polygon must be re-drawn, so that the vectors follow in the order of the points.

EXAMPLE. PQ (Fig. 283) *is a horizontal beam of length* 25·3 *ft., it is pin-jointed at* P, *and rests on a knife edge at* R, *and is partly supported by a rope fastened to it at* Q. *The rope* QU *passes over a* *smooth pulley at U, vertically above P, and a weight W is attached to the end. The beam being loaded at S, T and V, required to find the bending moment and shearing force at any point.*

$PQ=25{\cdot}3$ ft., $PU=12{\cdot}1$ ft., $PS=4{\cdot}36$ ft., $PT=10{\cdot}5$ ft., $PR=19{\cdot}7$ ft., $PV=22{\cdot}2$ ft.; the loads at S, T and V are 1650, 1890 and 1340 lbs. weight, and the weight suspended at the end of the rope is 3740 lbs.

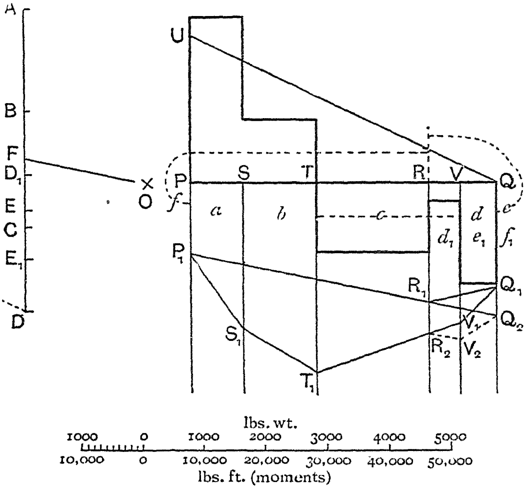

A
B
F
D_1
E
C
E_1
D
X
O
U
P
S
T
R
V
Q
f
a
b
c
d
e
d_1
e_1
f_1
P_1
R_1
Q_1
Q_2
S_1
T_1
V_1
R_2
V_2
lbs. wt.
1000
0
1000
2000
3000
4000
5000
10,000
0
10,000
20,000
30,000
40,000
50,000
lbs. ft. (moments)

Fig. 283.

Draw the vectors of the loads at S, T and V, viz. **AB**, **BC** and **CD**; then DE_2 (E_2 is not shewn in Fig.) parallel to QU for the tension in the rope. Project horizontally E_2 to E on AD. Notice that the space c in the beam diagram must go from T to V. Take the pole O of the vector polygon 10 units of length from AB. Draw the link polygon P_1S_1 parallel to OA, S_1T_1 parallel to OB, T_1V_1 (through whole space c) parallel to OC, V_1Q_1 parallel to OD, Q_1R_1 parallel to OE (so that the space e must be considered as going from QQ_1 round the top of the beam to the vertical through R). Close the link polygon by P_1R_1 and draw OF in vector polygon parallel to it. Then **EF** is the reaction at R, and **FA** the vertical component of the reaction at P. (The space f must be considered as going round from PP_1 over the beam to the vertical through R.)

The vertical intercepts of $P_1S_1T_1V_1Q_1R_1$ give the bending moments in lbs. ft. when measured on the force scale and multiplied by 10.

For the S.F. diagram set downwards from PQ at Q a distance $= DE$, at V a distance $= CE$, at R a distance $= FC$; at T set upwards a distance $= BF$ and at S, at distance AF, complete the rectangles as in the figure. The vertical intercept at any point between PQ and the thick horizontal lines gives the S.F. at that point.

As regards the B.M.; the fact that D_1R_1 is (if we start with OF in the vector polygon) the first and the last line of the link polygon, and yet for the space RQ we measure intercept from R_1Q_1 is perhaps a little difficulty. The difficulty is due to the links not being drawn in the order of the points. When we come to the right of R the intercept between P_1R_1 and T_1V_1 does not take account of the B.M. due to the reaction at R, but the intercept between R_1Q_1 and P_1R_1 (produced) does so, and, since, coming backwards from the right, Q_1R_1 is before P_1R_1, the intercepts have to be added.

It is, however, clearer to redraw the latter part of the link polygon by taking the forces in order. Letter the spaces between R and V, V and Q, d_1 and e_1, respectively, and the space c will now end at R.

In the vector polygon, from C draw CD_1 upwards, equal to the reaction at R, viz. EF; from D_1 set D_1E_1 downwards, equal to the load at V $(=CD)$; then E_1F is the vertical reaction at Q. The vector polygon is now FA, AB, BC, CD_1, D_1E_1 and E_1F. The link polygon is as before up to the space d_1; T_1V_1 stops at R_2; the next link through the space d_1 is R_2V_2 parallel to OD_1; and then V_2Q_2 through e_1 is parallel to OE_1. P_1Q_2 should be the same line as P_1R_1 if the construction is accurate. The B.M. diagram is now $P_1S_1T_1R_2V_2Q_2P_1$.

What are the B.M. and S.F. at distances of 12 and 21 ft. from P?

(6) The span of a roof truss is 40 ft.: three equal loads are placed at equal intervals of 10 ft., each load being 1·7 tons weight. The resultant wind pressure on the roof is equal to a force of 1 ton, and makes an angle of 45° with the horizontal, and its line of action passes through the mid-point of the line joining the points of support. The roof being supposed pinned at the end facing the wind and freely supported at the other, draw the diagram of the B.M. and S.F. for the forces perpendicular to the line joining the points of support.

(7) Find the B.M. and S.F. diagrams for the vertical post of the derrick crane in Fig. 284, due to a load W (3 tons) suspended at A. Length of jib $AC=16·8$ ft., length of tie rod $AB=14$ ft., $BC=10$ ft., $BE=20$ ft. The post is kept vertical by a smooth collar at D and a cup-shaped socket at E, and $DE=4$ ft.

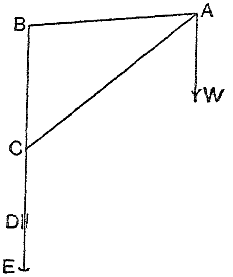


FIG. 284.

(8) Find the B.M. and S.F. diagrams if the chain supporting W is carried over a smooth pulley at A, and is fastened to the post at F the mid-point of BC, and the collar at D is replaced by a tie rod at D_1, sloping downwards at an incline of 30°, $ED_1=12$ ft.

(9) Find the B.M. diagram in Ex. (8) if the load is suspended from a point A_1 in BA produced, where $BA_1=17$ ft.

Beam uniformly Loaded.

EXAMPLE. *A beam 20 ft. long is uniformly loaded with 50 lbs. per foot run; draw the shearing force and bending moment diagrams, the beam being supported at its ends.*

Draw a line PQ, 20 cms. long, to represent the beam; draw a vertical upwards from the beam 1″ long to represent the load per ft. run. Complete the rectangle of base 20 cms. and height 1″. The area of this represents a load of 20×50 lbs. weight. Divide the rectangle into ten equal parts. Suppose the load on each of these parts concentrated at its M.C.

The reaction at each end is 500 lbs. weight.

Draw the link polygon for loads each of 100 lbs. weight concentrated at the M.C.'s of the rectangles.

Then draw the B.M. diagram. This diagram gives only approximately the B.M. at the various points, because the real loading is uniform and not ten equal detached loads. But, the vertices of the diagram are points on the true B.M. diagram. For consider any point X on one of the M.C. lines on the beam, the bending moment there = moment of reaction at $P - \Sigma$ moments of all the weights along PX. This last quantity is equal to the weight of PX multiplied by the distance of its M.C. from X, which is the sum of the moments of the partial system; and this difference is exactly what the B.M. diagram does give. On the other hand, for points between two of the M.C. lines, the diagram is wrong, since it neglects the load between them. The true B.M. diagram is a curve passing through all the vertices of the constructed diagram. Draw a smooth curve through the vertices and measure to scale in lbs. ft. the B.M. at points distant 3, 11 and 15 ft. from one end of the beam.

To draw the shearing force diagram. At Q, the right-hand end point of the beam, set downwards QQ_1 representing 500 lbs. to the proper scale. Join Q_1 to the mid-point of the beam and produce it to cut the vertical through P. This line, with the datum line PQ, forms the shearing force diagram; for the S.F. must decrease uniformly from 500 lbs. weight at Q to zero at the centre.

Beam continuously but not uniformly loaded. The method of the previous section applies to this case also.

EXAMPLE. *A horizontal beam* PQ (Fig. 285) *supported at the ends is continuously loaded, the load per foot run at any point being given by the ordinates of the triangle* PQC. *Find the* S.F. *and* B.M. *diagrams—the scale of the figure being horizontally* 1″ *to* 100 *ft., and vertically* 1 *cm. to* 0·5 *tons per ft. run.*

Divide the load curve into eight equal parts; find the vertical M.C. line of each part, and the load represented by the area of each part. Set these off to scale and draw the vector and link polygons as usual. Draw a curve through the vertices of the link polygon—this curve will be approximately the B.M. diagram.

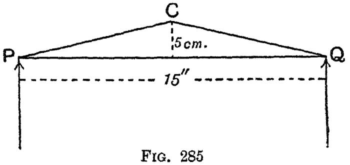


FIG. 285

Draw ordinates for the shearing forces at the end points of the sections from right to left, and draw a smooth curve through their end points.

(Remember that the S.F. diagram is such that the ordinate at any point gives the sum of the forces on one side of the beam.)

(10) Draw the B.M. and S.F. diagrams for a horizontal beam fixed in a vertical wall and projecting 25 ft., the load being uniform and 500 lbs. per ft. run.

(11) Draw the B.M. and S.F. diagrams for a cantilever due to its own weight and a load of 3 tons at its end, the length of the cantilever being 30 ft. and its weight 100 lbs. per ft. run. (Draw the diagrams separately and add the ordinate.)

(12) The length of a beam is given by PQ (7″) (scale 1″ to 8′) the load per foot run is given by the ordinates from PQ to a circular arc on PQ, the maximum ordinate being 4 cms. (scale 1 cm. to 0·5 ton per ft. run).
Draw the B.M. and S.F. diagrams and measure the B.M. and S.F. at points distant 21 and 15·3 ft. from the centre.

(13) Draw a right-angled triangle ABC having BC horizontal and of length 6·72″, and BA vertical of length 4·3″. Let BC represent the length of a cantilever from the fixed end B to the free end C, scale 1″ to 10 ft. Let the ordinate of the triangle at any point M of BC represent, to the scale of 1 cm. to 100 lbs. weight, the load per foot run there. Draw the diagrams of the bending moment and shearing force.

Travelling Loads. We have now to consider how the B.M. and S.F. at any point of a beam, bridge, or cantilever changes as one or more loads travel along it. Many structures of large span are now made with cantilevers connected by comparatively short girders; perhaps the best example of this kind of bridge is seen in the Forth railway. The whole bridge consists of two spans of about 1700 ft. each, two of 675 ft. each fifteen of 168 each and five of 25 each. For the main spans there are three double cantilevers, like scale beams, on supporting piers, and these cantilevers are connected by girders each 350 ft. long; the length of the double cantilever is 1360. For such massive cantilevers as those of the Forth bridge, the B.M and S.F. due to the travelling train are small compared with those due to the weight of the structure itself.

As the B.M. diagram is more easily constructed for a cantilever than for a girder, it is advisable to commence with the consideration of the former.

B.M. and S.F. Diagrams for a Travelling Load on a Cantilever.

EXAMPLE. *A cantilever is of length* 250 *ft. from pier to free end. Required diagrams giving the* B.M. *and* S.F. *at any point as a load of* 17·3 *tons travels from the free end towards the supported pier.*

In Fig. 286 PQ represents the length of the cantilever, Q being the free end, L the given point at a distance of 87 ft. from P.

AB is the load vector. Suppose the load at Q. Choose a pole O at a convenient distance from AB. Through R, a point on the vertical through L, draw RQ_1 and RQ_2 parallel to OA and OB. Then Q_1Q_2 gives the moment at L of the load at Q. When the load is at S, the intercept S_1S_2 on the vertical through S gives the B.M. at L; hence, RQ_1Q_2 gives the B.M. at L for all positions of the load between L and Q.

When the load passes L the B.M. vanishes.

The shearing force at L is constant for all positions of the load between L and Q; when the load passes L, the S.F. vanishes.

At all points, therefore, the maximum S.F. is the same and is equal to the load.

To find the B.M.'s for points other than L we have only to notice that moving L to the left is equivalent to moving Q an equal distance to the right. Through any point R on the vertical through P, draw RP_1 and RP_2, parallel to OA and OB

and cutting the vertical through Q in P_1 and P_2. The triangle P_1RP_2 gives the B.M.'s at P as the load travels from Q up to P. For a point L, distant x to the right of P, mark a point L', x to the left of Q, and draw the vertical $L_1L'L_2$; then L_1RL_2 is the B.M. diagram for L as the load moves from Q up to L (*i.e.* in new figure from L' to P).

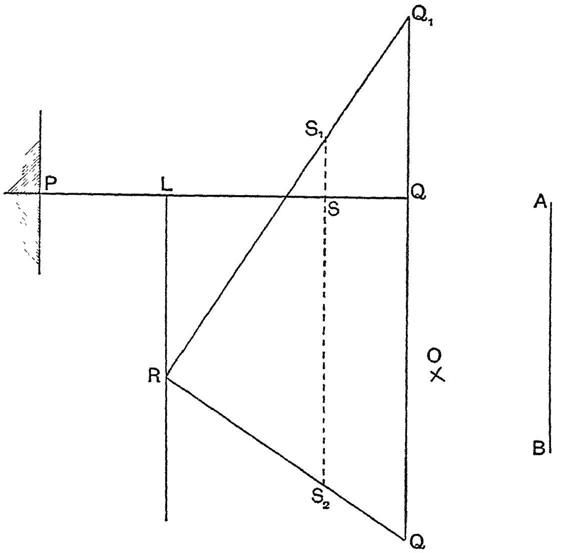


FIG. 286.

(14) What are the B.M.'s at points distant 30, 40, 100 and 150 ft. from P when the load is 30, 50, 80 and 100 ft. from Q.

(15) Find the maximum bending moment at ten points between Q and P. Set up, at these points, ordinates giving the maximum bending moments to scale, and thus find the curve of maximum B.M.'s.

*B.M. Diagram for Several Travelling Loads on a Cantilever.

EXAMPLE. *Loads of* 3, 5·2, 4·7 *and* 3·3 *tons weight travel along a cantilever of length* 57 *ft. The distances apart of the loads being* 5·3, 7·7 *and* 5·3 *ft. from the foremost load of* 3 *tons backwards, determine the* B.M. *and* S.F. *at any point as the loads travel from the free end up to the supporting pier or wall.*

Draw a line PQ (Fig. 287) to represent the length of the cantilever having Q for the free end, and mark the point L on it where the B.M. is required. PL represents 15 ft. Choose a pole O at, say, 10 units of distance from the load vectors **AB** ... **E**. Draw the axis ab of the load **AB** in its farthest position from L, and then on the other side of L draw axes b_1c_1, c_1d_1 and d_1e_1 at the proper distance of the loads apart from the axis through L (*i.e.* draw the axes as if the leading load was at L, and the loads were travelling towards Q).

Through any point A_1 on the axis ab_1 draw A_1X_1 parallel to AO, and A_1X_2 parallel to BO. Produce the latter line backwards to cut b_1c_1 in B_1; from B_1 draw B_1X_3 parallel to CO; produce this line backwards to cut c_1d_1 in C_1 and draw C_1X_4 parallel to OD; produce this line backwards to cut d_1e_1 in D_1, and draw D_1X_5 parallel to OE.

Then $X_1X_5D_1C_1B_1A_1$ is the B.M. diagram for L as the leading load travels from $M_1(ab)$ up to and past L, and the last load comes up to L (the first load being then at M_4).

When the leading load is at M_1, the B.M. is given by X_1X_5; when at M_2 it is given by Y_1Y_5; when at M_3 by Z_3Z_5; and when at M_4 it is zero.

Proof. Since $A_1X_1X_2$ is similar to OAB, the vertical intercepts of the former give (p. 322) the B.M. at L due to the first load as it travels from M_1 up to L

Again, $B_1X_2X_3$ is similar to OBC; and since the distance from b_1c_1 to ab is equal to that between L and the second load when the first load is at M_1, the vertical intercepts of $B_1X_2X_3$ must give the B.M. at L due to the second load, as it travels from bc

up to L. A similar argument shews that $C_1X_3X_4$ and $D_1X_4X_5$ give the B.M. due to the third and fourth loads, as these loads travel from their initial positions up to L.

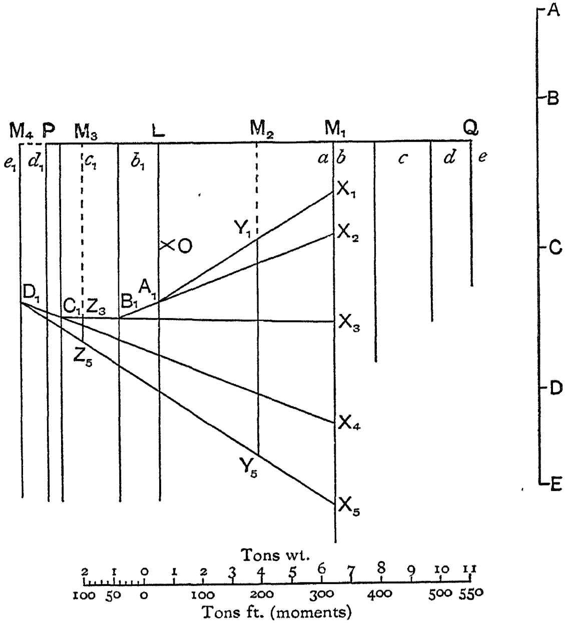

A
B
C
D
E
M4
P
M3
L
M2
M1
Q
e1
d1
c1
b1
a
b
c
d
e
X1
X2
X3
X4
X5
O
Y1
A1
B1
C1
Z3
D1
Z5
Y5
Tons wt.
2 1 0 1 2 3 4 5 6 7 8 9 10 11
100 50 0 100 200 300 400 500 550
Tons ft. (moments)

FIG. 287.

Since on passing L the load ceases to have any B.M. at L, the sum of the vertical intercepts of these triangles must give the total B.M. at L as the loads travel.

(16) Find the bending moment in tons ft. when the leading load is at (1) 21·5, (2) 11·8 ft. from L.

(17) Shew how to find the B.M.'s at L as the leading load travels from Q to M_1 (the loads may be supposed travelling from a second cantilever over a connecting girder to the one under consideration). Find the B.M. in tons ft. when the leading load is 7·2, 14·1 and 16 ft. from Q respectively.

To find the B.M.'s at points other than L it is not necessary to draw a fresh B.M. curve, because the lines A_1X_1, B_1X_2 ... are all fixed relatively to one another; and, therefore, instead of moving

L, it is sufficient to move the line X_1X_5. Thus, supposing the B.M. diagram for the travelling loads are required for a point 10 ft. to the right of L, then X_1X_5 must be drawn 10 ft. to the left of M_1; if for a point 10 ft. to the left of L, X_1X_5 must be drawn 10 ft. to the right of M_1, and the lines A_1X_1 ... produced to cut it.

The matter is thus mainly one of relettering the diagram originally drawn. Change L to P (the fixed end of the cantilever) and move M_1 to the right a distance LP, and change the letter to P'. Produce all the lines A_1X_1, B_1X_2 ... D_1X_5 to cut the vertical through P' in P_1P_2 ... P_5. Then $P_1A_1B_1$ ... P_5 is the B.M. diagram for P as the last load moves from the free end up to the fixed end. For any point L, distant x ft. to the right of P, draw a vertical through L', x ft. to left of P' and cutting A_1X_1, B_1Y_2 ... in L_1L_2 ... L_5; then $L_1A_1B_1C_1D_1L_5$ is the B.M. for L (at P) as the last load travels from the free end up to L.

(18) Find the maximum B.M.'s at points distant 5, 10, 15, 20, 25, 30, 35, 40 and 45 ft. from the pier, and draw the maximum B.M. curve.

(19) Draw the shearing force diagram for the point L as the loads travel up to and past the point.

(Up to the leading load being at L the S.F. is AE; immediately it passes L, the S.F. drops to BE and so on.)

(20) Loads of 10 tons 17·5 cwts., 10 tons 17·5 cwts., 19 tons, 19 tons, 12 tons, 12 tons 5 cwts. and 12 tons 15 cwts., due to an engine and tender, travel from a girder over a cantilever of length 150 ft. (from free end to pier). The distance apart of the loads from the leading one backwards are 6′ 6″, 8′ 9″, 10′, 8′ 7·25″, 6′ 9″ and 6′ 9″ respectively

Draw the diagram of the B.M. for a point distant 25 feet from the pier as the engine and tender travel over the cantilever.

(21) Alter the lettering so that the diagram will give the B.M's. at any point of the cantilever, and determine the B.M. at points distant 50 and 75 ft. from the pier when the leading wheels of the engine are at 110, 100, 90, 75, 50, 10 and 5 ft. from the pier.

Travelling Loads on a freely supported Bridge.

EXAMPLE. AB (Fig. 288) *is a beam freely supported at its ends. A load* W *travels from* A *to* B; *required the* B.M. *at any point* Q *for every position* P *of the load* W.

The reactions R_1 and R_2 at A and B, due to W at P, are given by

$$R_1 \, . \, AB = W \, . \, PB \text{ and } R_2 \, . \, AB = W \, . \, AP,$$

and the bending moment at Q is

$$R_2 \,.\, QB = \frac{W}{AB} \,.\, AP \,.\, QB.$$

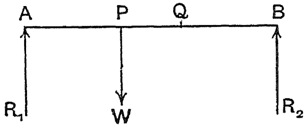


FIG. 288.

Now suppose W at Q, then the reactions R_1' and R_2' are given by

$$R_1' \,.\, AB = W \,.\, AB \text{ and } R_2' \,.\, AB = W \,.\, AQ,$$

and the bending moment at P is

$$R_1' \,.\, AP = \frac{W}{AB} \,.\, AP \,.\, QB.$$

Hence *the* B.M. *at* Q *due to the load* W *at* P *is the same as the* B.M. *at* P *due to the load* W *at* Q.

B.M. and S.F. Diagrams for a Travelling Load on a freely supported Bridge.

EXAMPLE. *A horizontal beam* AB (Fig. 289), *of length* 50 *ft.*, *is freely supported at its ends; a load of* 3·38 *tons travels from* A *to* B; *required diagrams giving the* B.M. *and* S.F. *at a point* Q (QB = 16·4 *ft.*) *for all positions of the load.*

Draw the B.M. diagram for a load of 3·38 tons at Q, and through any point P in AB draw a vertical PP_2P_3 cutting the B.M. curve at P_2 and P_3. Then the intercept P_2P_3 gives, to scale, the B.M. at Q due to the load 3·38 tons at P.

Set upwards from A, to scale, AC = the load, join BC cutting the vertical at Q in Q_1. From A draw AQ_2 parallel to BQ_1 cutting the vertical at Q in Q_2. Then AQ_2Q_1B is the S.F. diagram with AB as base line.

The vertical at P cuts the diagram at P_1, and PP_1 measures the S.F. at Q due to the load at P.

Proof. That for the B.M. has already been given.

For the S.F. we notice that when the load at P is between A and Q the S.F. at Q is the reaction at B, and when P is between Q and B the S.F. at Q is the reaction at A.

If R_1 is the reaction at A when the load is at some point P' between Q and B, then $R_1 . AB = W . P'B$

$$\therefore \text{ S.F. at } Q = R_1 = W\frac{P'B}{AB} = W\frac{P'P_1'}{AC},$$

hence $P'P_1'$ measures to scale the S.F. at Q, and therefore the S.F. at Q, when the load is between Q and B, is given by the ordinate of the diagram BQQ_1.

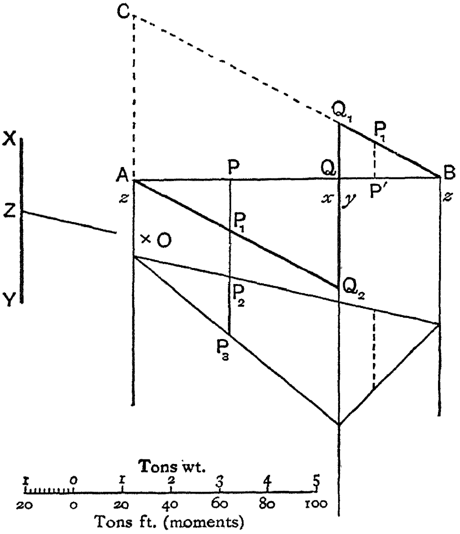

C
Q₁
P₁
X
A
P
Q
B
z
x
y
P′
z
Z
P₁
×O
Q₂
Y
P₂
P₃
Tons wt.
1 0 1 2 3 4 5
20 0 20 40 60 80 100
Tons ft. (moments)

FIG. 289.

Similarly, if the load is at P between A and Q, and R_2 is the reaction at B, $R_2 . AB = W . AP$;

$$\therefore \text{ S.F. at } Q = R_2 = W\frac{AP}{AB} = W\frac{PP_1}{AC}, \text{ (since } AQ_2 \text{ is parallel to } BC\text{)};$$

therefore the S.F. at Q, when the load is between A and Q, is given by the ordinate of the diagram AQ_2Q.

Immediately before P comes up to Q the S.F. is QQ_2, and immediately after it is QQ_1.

The maximum S.F. is therefore immediately before the load, travelling from A, comes up to Q.

The maximum B.M. is when the load is at Q.

(22) What are the B.M. and S.F. at Q when the load is at P and
(i) $AP=25{\cdot}7$ ft., (ii) $AP=41{\cdot}6$ ft.

Curve of Maximum Bending Moment for a Travelling Load. To find the maximum bending moment for other positions of Q, it is not necessary to redraw the whole bending moment diagram. A new closing line only is wanted.

Keeping the pole O fixed, the lines R_1R_2 and R_2R_3 (Fig. 290), being parallel to OX and OY, must always have the same directions, and hence, instead of supposing Q moved relatively to A and B, we may suppose A and B moved relatively to Q.

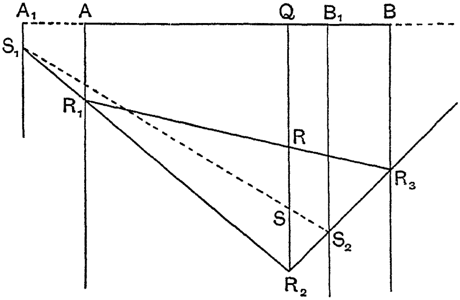


FIG. 290.

Suppose the maximum bending moment were required at a distance x from A. Set off $QA_1=x$ from Q towards A and make $A_1B_1=AB$. Mark the points S_1 and S_3, where R_1R_2 and R_2R_3 cut the verticals through A_1 and B_1, join S_1S_3 cutting the vertical through Q in S. Then SR_2 gives the maximum B.M. at a distance x from A.

(23) Find the maximum bending moment due to the load at points distant 5, 10, 15, 20, 25, .., 45 ft. from A (Fig. 289). Set up at these points ordinates giving the maximum B.M.'s to scale and join them by a smooth curve, which must evidently pass through A and B This curve is the maximum B.M. curve for a single travelling load.

*B.M. due to several Travelling Loads.

EXAMPLE. *To find the* B.M. *at any point of a bridge, freely supported at its ends, as an engine travels from one end (abutment) to the other.*

A bridge is 50 *ft. long and an engine travels over it; draw a diagram giving the* B.M. *at the mid-point of the beam for all positions of the engine whilst wholly on the bridge. The load on the leading wheels is* 17 *tons* 18 *cwts., and on the following ones* 19 *tons* 16 *cwts.,* 19 *tons* 16 *cwts. and* 17 *tons respectively. The distances between the centre lines of the wheels are, from rear to front,* 8′ 3″, 7′ 0″ *and* 9′ 0″ *respectively.*

Draw the bridge length, XY (Fig. 291), to scale and mark the load lines when the engine is in one definite position, say with the centre of the trailing wheels 2·7′ from the left abutment. Draw the load vectors **ABCDE** and take a pole P at a convenient distance. Construct the link polygon $R_0R_1R_2R_3R_4R_5$ in the usual way, close it by the link R_0R_5, and draw in the vector polygon PO parallel to R_0R_5. Mark the point R where the first and last lines of the link polygon intersect. Bisect R_0R_5 at M; mark the point M_5 on RR_5 so that the horizontal distance between R and M_5 is half the span (25 ft.); join M and M_5 and produce both ways.

From the line ab mark off to the right horizontally a distance $= \frac{1}{2}$ span and determine a point L; similarly, from de mark off horizontally to the left the same distance and determine a point K; draw verticals through K and L; then the figure between these verticals, MM_5 and $R_1R_2 \dots R_5$, is the bending moment diagram for the mid-point of the beam, from the trailing wheel leaving X to the leading wheel reaching Y.

Maximum Bending Moment at the Centre. An inspection of Fig. 291 shews that M_2R_2 is the greatest value of the B.M. and this occurs when the third wheel of the engine is just over the mid-point of the bridge.

When the leading wheel is over the mid-point, the B.M. is given by M_4R_4; when the trailing wheel is over the mid-point the B.M. is given by M_1R_1; and so for all positions of the engine.

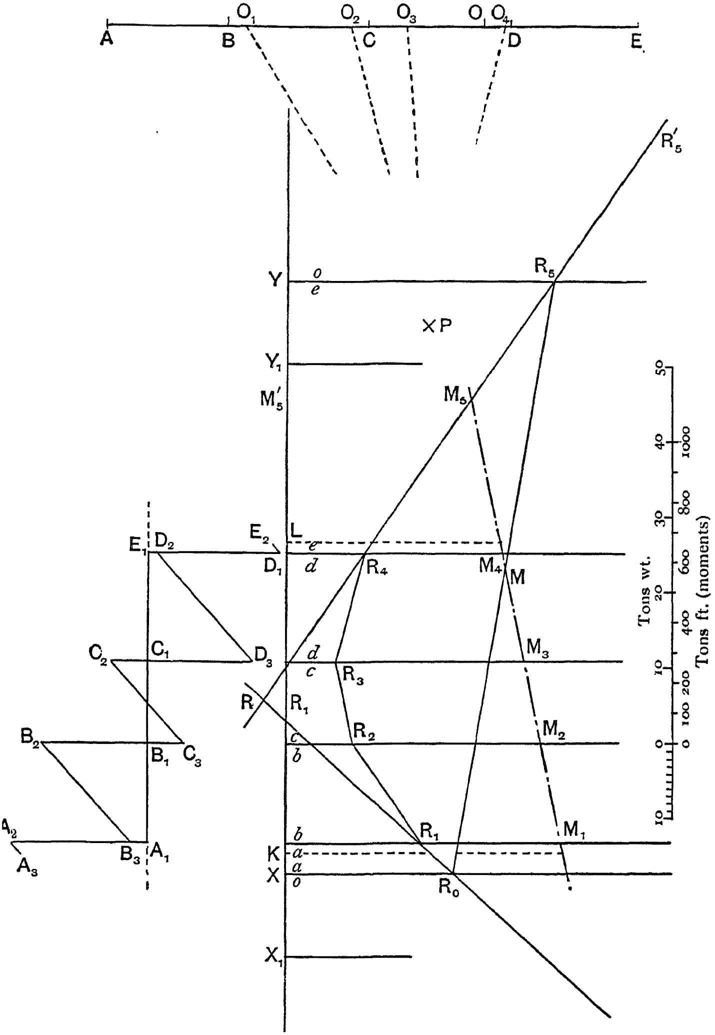

A
B
C
D
E
O₁
O₂
O₃
O
O₄
Y
Y₁
M′₅
E₁
D₂
E₂
L
D₁
C₂
C₁
D₃
R
R₁
B₂
B₁
C₃
A₂
A₃
B₃
A₁
K
X
X₁
o
e
d
c
b
a
XP
R₅
R′₅
M₅
R₄
M₄
M
M₃
R₃
R₂
M₂
R₁
M₁
R₀
Tons wt.
Tons ft. (moments)
50
40
30
20
10
0
10
1000
800
600
400
200
100
0

FIG. 291.

(24) Find the B.M. diagram for a point distant 15 ft. from the left abutment.

Mark the point on XY, project vertically to R_0R_5 cutting it at S, from R go horizontally 35′ and then vertically to S_5 on RR_5. Join SS_5; then the vertical distances between SS_5 and $R_0R_1 \dots R_5$ give the B.M. at the point required. The horizontal distances between which this will hold are 15 ft. to the right of *ab* and 35 ft. to the left of *de*.

(25) What is the maximum B.M. at this point?

If the pole P be kept fixed, then, whatever the position of the engine on the bridge, the link polygon $R_1R_2R_3R_4$ and the lines RR_1 and RR_4 will be in exactly the same relative positions. The only things that alter as the engine moves are the reactions at the ends, which alter the direction of the closing line R_0R_5.

Instead, therefore, of supposing the loads to move, we may suppose the supports moved. Thus, to get the second pair of wheels *cd* over a point Q (distant x from X) of the bridge, we have only to set off x to the left and $50-x$ to the right of *cd*, and the points so determined are the new position X_1Y_1 of the supports. The verticals through X_1 and Y_1 cut R_0R and RR_4 in, say, G_0 and G_5, and G_0G_5 is the closing line. G_0G_5 cuts *cd* in G_3, say, then G_3R_3 gives the B.M. at *cd* (which is now the vertical through Q).

Now G_3 divides G_0G_5 in the ratio $x : 50-x$. If, therefore, we could find the locus of the points dividing the closing chords in this ratio, we could read off at once the vertical distances between the locus and $R_1R_2 \dots R_5$, giving the B.M. at the point required as the engine travels along the bridge. *This locus is a straight line.*

(26) Find the closing lines of the B.M. diagrams for points distant 10, 20, 30 and 40 ft from X, and measure the maximum B.M. at those points.

(27) Set up ordinates at points along XY corresponding to the maximum B.M.'s determined, and join the end points by a straight line. It is the maximum B.M. diagram for all points on the bridge, as the engine, being wholly on the bridge, travels from left to right.

***A straight line of variable length moves so that its end points describe straight lines, the ratio of the distances moved through by these end points being constant, the locus of the point dividing the moving line in a constant ratio is a straight line.**

R_0R_5 and G_0G_5 (Fig. 292) are any two positions of the moving line.

As G_0 and G_5 move $\frac{G_0R_0}{G_5R_5}$ is always the same.

S divides R_0R_5 in a given ratio.

Join G_0R_5, and draw SS_1 parallel to G_0R_0 cutting G_0R_5 in S_1, then draw S_1S_2 parallel to R_5G_5 cutting G_0G_5 in S_2.

Then, since $\frac{R_0S}{SR_5}=\frac{G_0S_1}{S_1R_5}=\frac{G_0S_2}{S_2G_5}$,

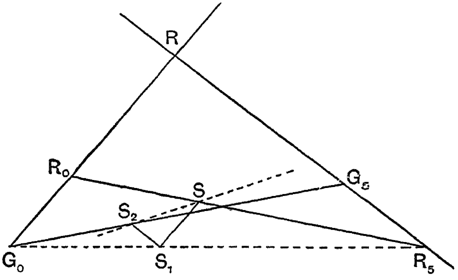


FIG. 292.

S_2 must divide G_0G_5 in the given ratio.

But SS_1 bears a fixed ratio to G_0R_0,
and S_1S_2 „ „ G_5R_5;

$$\therefore \frac{S_1S}{S_1S_2} \text{ is constant if } \frac{G_0R_0}{G_5R_5} \text{ is constant.}$$

Also the angle S_2S_1S is equal to the angle at R, and is therefore constant. Hence, whatever the position of S_1 and S_2, the triangle SS_1S_2 retains the same shape; the angle S_2SS_1 is therefore constant; and since SS_1 is always parallel to R_0R, S_2 lies in a fixed direction from S, *i.e.* the locus of S_2 is a straight line through S.

Referring back to the B.M. diagram, we see that this locus may be drawn by finding out where it cuts R_0R or R_5R, and joining the point so determined to the given point in R_0R_5.

When G_0 comes to R, then G_5 must be 50 ft. horizontally to the right of R, and hence where the locus cuts RR_5 is determined by dividing RG_5 in the given ratio. Since, for the special case drawn, the ratio is unity, S is at M and S_2 is at M_5, where the horizontal distance between R and M_5 is 25 ft.

Similarly, if the point for which we want the B.M. is at a distance x from the left-hand abutment, then, when G_0 comes to R, G_5 must be 50 ft. horizontally from R, and the point required must be x ft. from R (horizontally); hence set off x ft. horizontally from R, and project vertically down to RR_5.

As regards the limits within which the straight line locus is available, we must remember that unless all the loads are on the bridge, the link polygon will not be the same. When the leading wheels come to the right-hand abutment, *i.e.* when the right-hand abutment is at *de*, the point distant x from the left-hand abutment will be $50 - x$ from *de*, and when the last pair of wheels is just on the left-hand abutment (*i.e.* when the left-hand abutment is at *ab*) the point required is at x ft. to the right of *ab*.

When only part of the train is on the bridge, only the part of the link polygon corresponding to the load actually on the bridge must be taken. Thus, if the leading wheels have passed the right-hand abutment, R_3R_4 is the last line of the vector polygon, and the line joining R_0 to the intersection of R_3R_4 and *eo* is the closing line. Similarly, if the first two pairs of wheels have passed the right-hand abutment, R_2R_3 is the last link, and the line joining R_0 and the point of intersection of R_2R_3 and *eo* is the closing line.

By dividing these closing lines in the ratio x to $50 - x$, points on the locus from which the B.M.'s are measured may be found for the various cases when all the engine is not on the bridge.

***S.F. Diagram for more than one Travelling Load.** The S.F. is determined at any point when we know the closing line of the link polygon.

Take the case of the S.F. at the mid-point.

When the leading wheel is just coming up to the mid-point, M_4 is a point on the closing line (Fig. 291); draw then in the vector polygon PO_4 parallel to the closing line through M_4 (this closing line cuts RR_0 at a point distant 25′ horizontally from M_4). The S.F. is then O_4E; immediately the leading wheel passes the mid-point, the S.F. drops to O_4D.

When the second wheel is over the mid-point, M_3 is on the closing line; draw then PO_3 parallel to this line. Just before the second wheel reaches the mid-point, the S.F. is O_3D; and just after passing it the S.F. is O_3C and has changed sign.

Similarly, when the third wheel is just coming to and just going from the mid-point, the S.F.'s are O_2C and O_2B, and when the fourth wheel is just going to and from the mid-points, the S.F.'s are O_1B and O_1A respectively.

Draw a horizontal datum line A_1E_1 for the S.F. perpendicular to the load lines *ab*, *bc*, *cd* and *de*. At E_1 (on *de*) set downwards E_1D_1 and E_1D_2 equal to O_4E and O_4D respectively, at C_1 (on *cd*) set downwards $C_1D_3=O_3D$, and upwards $C_1C_2=O_3C$. At B_1 (on *bc*) set downwards O_2C, and upwards O_2B. At A_1 (on *ab*) set up $A_1B_3=O_1B$, and $A_1A_2=O_1A$.

Then $E_2D_1D_2B_3C_2C_3B_2B_3A_2A_3$ is the shearing force diagram for the mid-point of the bridge as the engine travels over the bridge from left to right, from the moment when the trailing wheel is on the left-hand abutment to the moment when the leading wheel is on the right-hand abutment.

The maximum value of the S.F. at the mid-point is seen from Fig. 291 to be when the trailing wheel has just passed the mid-point.

MISCELLANEOUS EXAMPLES. IX.

1. A beam 30 ft. long, supported at the ends and weighing 1000 lbs., carries a load of 1500 lbs. 10 ft. from one end. Shew how to find the moment of the force tending to bend the beam at any point; shew in a graph this moment for all points of the beam and find where the beam is likeliest to break. (Home Civil, I., 1905.)

2. A beam 40 ft. long is loaded with three weights of 5, 15 and 10 tons placed 10 ft. apart, the 15 ton weight being at the centre of the span. Draw the diagram of the bending moments and the shearing stresses. (Admiralty, 1904.)

3. A beam is in equilibrium under any system of parallel forces, acting in a plane which passes through the axis of the beam. Explain what is meant by the shearing force and the bending moment at any section, and shew how to determine their values.

The beam AB is 20 ft. long, and rests horizontally on two supports, one at A, the other 5 ft. from B. There is a load of 2 tons midway between the supports, and a load of 1 ton at B. Draw a diagram for the bending moments along the beam. (Patent Office, 1905.)

4. Find the stress diagram of a triangular crane ABC, of which AB is the vertical post, AC a horizontal beam supported by a tie rod BC, due to a load of W tons carried at a point D of AC. Prove that the bending moment at D is $W\frac{AD \, . \, DC}{AC}$ ft.-tons. (Inter. Sci., 1902.)

5. Draw a rectangle $ABCD$ and its diagonals AC, BD intersecting at E, the lengths of AB and AC being 6 ft., and let its plane be vertical and AB horizontal. Let AC and BD represent two weightless rods, turning freely round a pin at E, with their lower ends A, B connected by a thread, and standing on a horizontal plane. If a weight is hung at C, find the pressure on the ground, the tension of the thread, the stress on the pin at E, and the stresses in the rods themselves.

How would the results be affected if C and D were connected by a thread instead of A and B? (B. of E., II., 1902.)

6. BC, CA, AB are three weightless rods formed into a triangular frame; their lengths are respectively 10, 8, 6; the frame is hung up by the angular point A; a weight of 100 lb. is hung from the middle point of BC. Find the stresses in BC.

Find also what difference it would make in the stresses if 50 lb. were hung at B and 50 lb. at C, instead of 100 lb. at the middle of BC.
(B. of E., II., 1903.)

7. Draw bending moment and shearing force diagrams for a beam loaded as follows:

A uniformly distributed load of 3 cwt. per foot run covers $\frac{2}{3}$ of the span from one abutment, and the span is 60 ft. Mark on your drawing the position and amount of the maximum bending moment.
(B. of E., III., Applied Mechanics, 1904.)

8. A bridge has a span of 72 ft. Draw the bending moment and shearing force diagrams for a point distant 9 ft. from the right-hand abutment as an engine travels from the left to the right-hand abutment. The distances apart of the centre lines of the wheels are 7′, 5′ 6″, 7′ and 8′ 3″ from the leading wheels backwards, whilst the loads on the wheels are 8 tons 19 cwts., 8 tons 19 cwts., 19 tons 16 cwts., 19 tons 16 cwts. and 17 tons 0 cwt. in the same order.

CHAPTER X.

STRESS DIAGRAMS (*Continued*).

In designing roof trusses, etc., the engineer has to take into account not only the permanent loads but also the pressures due to wind and snow.

Indeterminate Reactions. The wind pressure, being normal to the roof, has a horizontal component tending to slide the roof off its supports. This is, of course, resisted by the walls, but how much is borne by each wall it is generally impossible to say, since a given force may be resolved into two passing through fixed points (the points of support) in an infinite number of ways. This does not mean that the reactions of the supporting walls are indefinite, but simply that further information is necessary to determine them.

A similar difficulty occurs in the attempt to determine how much of the weight of a door is borne by each of the two hinges. Here the indeterminateness is due to not knowing the exact relation between the parts of the hinges screwed to the door and the parts screwed to the door post.

When the door is put into position, it may happen that the upper hinge parts only come into contact, and then the whole weight of the door is borne by the upper hinge; similarly for the lower hinge. Again, if the upper hinge parts come into contact first, the wood may give slightly, so that finally the lower hinge parts come into contact; in this case we must know a good deal about the elastic properties of the wood and hinge before the hinge loads can be determined.

The hinge parts on the post being at slightly different distances apart from the corresponding parts of the hinges on the door, the latter may have to be forced into position and any amount of compressive or tensile stress may thus be brought into play at the hinges. Finally, change of temperature, or accident, may alter the initial positions of the various parts.

Reactions made determinate. In heavy gates it is not uncommon to have one hinge only near the top, the lower being replaced by a vertical iron plate against which the lower part of the gate (or rather a rounded iron fixed to the gate) presses. The forces in this case are determinate, since the reaction of the plate must be horizontal.

A similar device is sometimes used for large roof trusses. The truss is hinged (pin-jointed) to one wall, the other end of the truss forms an iron shoe jointed to the axle of an iron roller, which rests on a horizontal iron plate on the top of the wall. The reaction of the plate on the truss is therefore vertical and the resultant of the external forces being known, the reaction at the hinge can be determined, and therefore the stresses in all the bars.

Reactions and Stresses due to Loads and Wind Pressure.

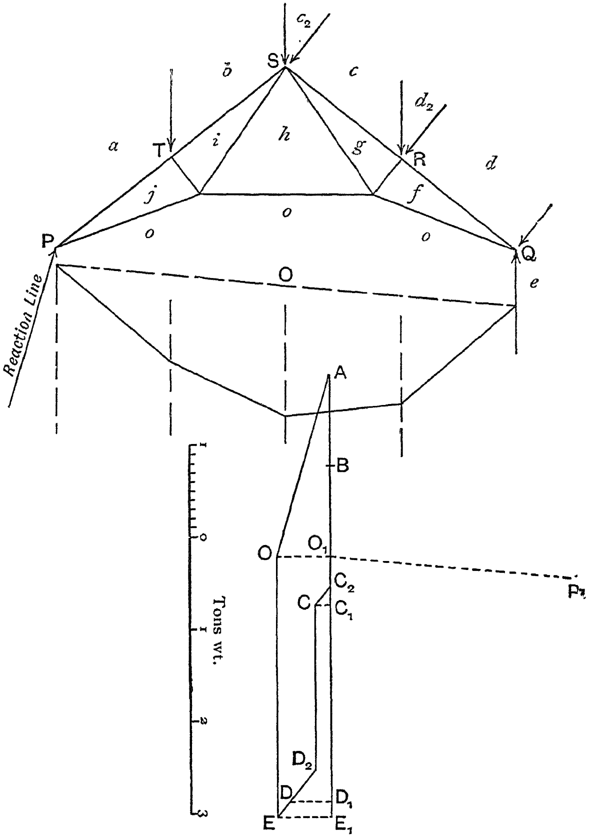
EXAMPLE. PQRST (Fig. 293) *represents a roof truss, pin-jointed at* P, *with an iron roller shoe at* Q. *The loads at* T, S, *and* R *due to the roofing and snow are* 1, 1·3 *and* 1·75 *tons. The wind pressure is equivalent to forces of* 0·25, 0·5 *and* 0·2 *tons at* S, R *and* Q *perpendicular to the roof. Determine the reactions at* P *and* Q, *and the stresses in the bars.*

Given $PQ = 30$ ft., $QR = RS = 19$ ft., and that the base and altitude of the central triangle are 11·3 ft. and 8 ft.

Draw the truss to scale and letter the spaces; then draw the vector polygon **AB, BC, CD, DE** of the forces to scale.

Project on to the vertical and obtain $ABC_1D_1E_1$; then $\mathbf{E_1E}$ is the horizontal force borne by P.

Take any pole P_1 of the vector polygon and draw the link polygon for the vertical forces **AB**, $\mathbf{BC_1}$, $\mathbf{C_1D_1}$. Determine the vertical reactions at P and Q due to these by closing the link polygon. In Fig. 293 these are $\mathbf{D_1O_1}$ and $\mathbf{O_1A}$. The total vertical reaction at Q is $\mathbf{E_1O_1}$. Draw OO_1 equal and parallel to EE_1, then **OA** is the total reaction at P, and **EO** that at Q.



c_2
S
b
c
d_2
a
T
i
h
g
R
d
j
f
o
P
o
o
Q
O
e
Reaction Line
A
B
O
O_1
C_2
C
C_1
P_1
Tons wt.
1
0
1
2
3
D_2
D
D_1
E
E_1

Fig. 293.

The vector polygon for the external forces is now *OABCDEO*.

Another way would be to letter the spaces between the vertical loads and the wind pressures as indicated, c_2 and d_2, and draw the link polygon for all the forces **AB**, $\mathbf{BC_2}$, $\mathbf{C_2C}$, $\mathbf{CD_2}$, $\mathbf{D_2D}$, **DE** and determine the axis of the resultant. Then find the point of intersection of this axis and the known line of reaction at *Q* and join *P* to this point. The reaction at *P* is thus determined in direction, and the forces at *P* and *Q* will be given in the vector polygon by drawing lines from *A* and *E* parallel to the reaction lines.

Draw now the stress diagrams for the bars in the usual way, and tabulate the results. Shew in the frame diagram those bars which are in compression.

(1) Find the stresses in the bars if *Q* be pin-jointed and *P* be on a roller. Tabulate the results.

(2) Find the stresses in the bars on the supposition that *P* and *Q* each bear half the horizontal thrust of the wind.

(3) Find the stresses on the supposition that *P* and *Q* are fixed and the wind pressures at *S* and *R* can be replaced by forces through *P* and *Q* parallel to them.

(Fixing *Q*, in addition to *P*, renders the frame over rigid, and merely putting the frame in position may set up large stresses. Again, suppose the temperature rises, then the bars tend to elongate whilst the fixed points *P* and *Q* resist these elongations. Hence, stresses will be set up quite independently of the loads. We are, therefore, driven to further assumption that the truss can be fixed in position without causing stress, and that the temperature does not change. These suppositions are sometimes made in books dealing with actual engineering structures, but there is no real justification for them, see also pp. 337 and 338.)

(4) Find the stresses in the bars of the queen post truss of Ex. 21, Chap. VI. (p. 225), if *Q* be pin-jointed, and *P* on rollers, and if the normal pressures due to the wind at *T*, *S*, *R*, *Q* be 0·25, 0·5, 0·5 and 0·25 tons weight.

Stresses found by Moments. On p. 248 was given a method of sections for finding the stresses in one or more particular bars by the resolution of forces into three components lying along three non-concurrent lines. A similar method of sections combined with the moment construction will also often give the stresses in particulars bars.

For this method to be effectual we must be able as before to make an ideal section of the frame cutting the particular bar for which stress is wanted, whilst the rest of the cut bars are concurrent.

A good example of this moment method is afforded by the suspension bridge problem already considered from another point of view in Chap. VI., pp. 242-248.

Suspension Bridges. In suspension bridges, the roadway is supported by two sets of *equidistant* vertical rods (tie-rods) which are attached, at their upper ends, to the pins of long linked chains supported on pillars at the ends of the bridge. The pillars are kept vertical either by passing the chains over their tops and fixing them to blocks in the ground, or by means of separate tie-rods (backstays).

In Fig. 284, PQ represents the roadway supported by eight tie-rods (only a few are taken for the sake of simplicity). PR and SQ are the supporting pillars, RT and SU the tie-rods keeping the pillars in a vertical position.

The problem is: *Given the span* PQ (60 *ft.*) *of the bridge and the dip of the chain,* i.e. *the vertical distance of the lowest point* L *from the highest* S (15 *ft.*), *to find the lengths of the various links, their slopes and the stresses in them.*

If the roadway be uniform, each vertical tie-bar bears an equal fraction of the total weight. Let this be w (3·5 tons).

Since the position of the chain cannot depend on the slope of the tie-rods, we may suppose these rods replaced by a light rigid strut joining RS. In this case PR and QS must react on R and S with equal forces of $4w$ (14 tons).

Number the spaces as indicated and set out the load vectors **AB, BC, CD, DE, EF,** .... From E draw EO perpendicular to AE, and make EO represent 10 ft. to the scale to which the span and dip were drawn.

Draw the link polygon for the pole O, starting with the first link, parallel to OA, through R. The middle link eo will evidently from symmetry be horizontal, but in all probability will not go through L.

Through any point Z in the lowest link draw the vertical XYZ cutting RS in X and the desired position of the lowest link in Y.

In the vector polygon draw OR perpendicular to YS, and RO_1 perpendicular to ZS. Then O_1 is the correct position of the pole and EO_1 is the stress in the lowest link.

This last construction is simply to find EO_1 so that $\frac{EO_1}{EO}=\frac{XZ}{XY}$ and this is done by making the sides of REO and REO_1 perpendicular to the sides of XYS and XZS.

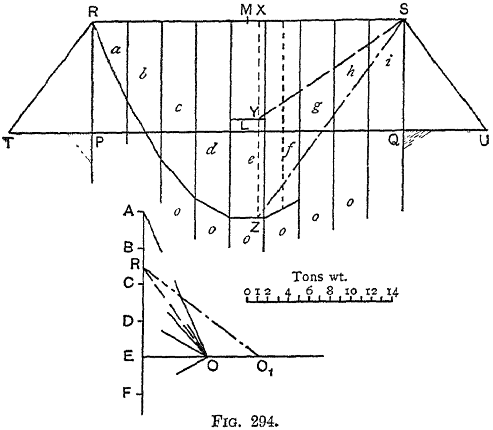


FIG. 294.

Draw the link polygon with O_1 as the pole of the vector polygon, and see that if the first link be drawn through R the middle one will pass through L.

Now draw to scale the supporting pillars RP and SQ, length, say, 18 ft., and the tie-rods RT and SU making 40° with the vertical. In the vector polygon AO_1 gives the tension in the link *oa*; hence draw through O_1 a line parallel to RT (link polygon) cutting AE in N; then NA is the reaction along PR and O_1N is the tension in RT.

Proof. If O be the pole giving Z for the lowest link, and O_1 the pole for Y, then $XY.EO_1$ measures the sum of the moments of all the forces to the right of XY about any point in XYZ (Chap. VII., p. 291). Similarly, $XZ \, . \, EO$ measures the sum of these moments, and, therefore, $XY \, . \, EO_1 = XZ \, . \, EO$. O_1 was determined from this equation; therefore EO_1 gives the stress in the link LY and the other links must be parallel to the corresponding lines of the vector polygon.

The position of O_1 may also be determined by calculation. The distances of the load lines from the middle point of the lowest link are all known; the reaction at S is also known, viz. half the sum of the loads. Hence, taking moments about the middle point of RS, we have the sum of the moments of the external forces = stress in the middle link multiplied by XY. This stress must be set off from E along EO on the force scale; it should come to O_1.

Notice also that the horizontal component of all the tensions is the same in magnitude, hence the forces in the cut bars at X and Y must form a couple of momental area equal in magnitude and opposite in sense to the couples formed by the reaction at S and the resultant of the loads *ef*, *fg*, *gh* and *hi*.

If the number of the rods be odd no link will be horizontal.

The construction for finding the pole of the vector polygon is still the same. Z must be taken on the middle load line, and O on the perpendicular through the mid-point of the resultant load vector.

(5) Draw the stress diagram for the same span, dip and load per vertical tie-rod if there be 9 vertical tie-rods and 10 spaces.

(6) The span is 100 metres, the dip 8 metres, and there are 12 vertical tie-rods each bearing a load of 20,000 kilogrammes. Draw the chain to scale.

The Suspension Bridge and Parabola. The connection between these was given in Chap. VI., p. 247. A much simpler proof by moments can now be given.

Let V (Fig. 295) denote the lowest vertex and Q the n^{th} one from V, so that between V and Q there are $n-1$ equal loads each of magnitude w. Since the loads are equidistant, the resultant load must be vertical and midway between V and Q. Take the horizontal and vertical through V as the axes of coordinates. Then if x and y are coordinates of Q, the resultant load is at a distance $\frac{x}{2}$ from V.

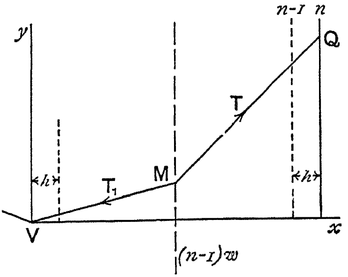


FIG. 295.

Let QM and VM be the direction of the links at Q and V; then since the chain between V and Q is in equilibrium under three forces, the total load and the tensions along VM and MQ, these must meet at a point on the resultant load, *i.e.* at M, whose abscissa is $\frac{1}{2}x$.

Take moments about Q, then the moment of the total load acting through M is equal to the moment of T_1 in VM; but T_1 is equivalent to T_0 horizontally and $\frac{w}{2}$ vertically, hence, taking account of sense,

$$w(n-1)\frac{x}{2} = T_0 y - \tfrac{1}{2}wx;$$

$$\therefore\ w\,.\,n\frac{x}{2} = T_0 y,$$

and if h = distance apart of tie rods

$$nh = x.$$

$$\therefore\ \frac{wx^2}{2h} = T_0 y \text{ or } y = \frac{w}{2T_0}\cdot\frac{x^2}{h}, \qquad \text{...... ....(i)}$$

a parabola.

T_0 is determined when the span and dip and number of spaces are known. Say span = 100 ft., dip = 20, $h = 5$,

then $$20 = \frac{w}{2T_0}\cdot\frac{(50)^2}{0{\cdot}5};$$

$$\therefore\ T_0 = 12{\cdot}5w.$$

If the middle link be horizontal (Fig. 296),

then $$MN = \frac{nh}{2},$$

and, taking moments about Q,

$$w(n-1)\frac{nh}{2}=T_0 y;$$

and since $$x-\frac{h}{2}=(n-1)h,$$

$$x+\frac{h}{2}=nh;$$

$$\therefore\ n(n-1)h^2=x^2-\frac{h^2}{4};$$

$$\therefore\ y=\frac{w}{2hT_0}\left(x^2-\frac{h^2}{4}\right) \quad \text{.........................(ii)}$$

as on p. 248.

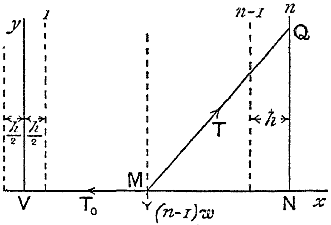


FIG. 296.

Links forming a continuous Curve. However large n may be (i) will be true always, but both h and w become smaller and smaller as the number of links is increased. When the loading is continuous, $\frac{w}{h}$ is the load per horizontal unit of length (or foot run), and the chain assumes a continuously curved form.

Such a loading and chain cannot be obtained easily but very near approximations are possible. Telegraph and telephone wires, for which the sag in the middle is small, are cases in point. The sag being small, the distance between the points of support is approximately the same as the length of the wire, and hence the load (which is continuously applied) may be taken as constant per ft. run.

If the wire or cable were of variable section, so that the weight per horizontal ft. run was constant, the above supposition would hold whatever the sag.

If $\frac{w}{h}=W$ then $y=\frac{W}{2T_0}x^2$.

The equation to the curve assumed by the chain may also be determined directly, as in the following article.

Uniformly loaded Chain. In this case the number of vertical tie-rods is supposed so great that the roadway may be regarded as being continuously supported. Let W = weight of roadway per unit length; then the weight of any length x is Wx, and the axis of the resultant weight acts through the mid-point of the length x.

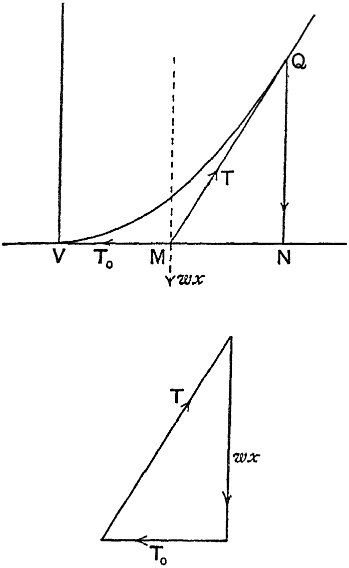


FIG. 297.

Let V (Fig. 297) be the lowest point of the chain, then the chain is horizontal at that point. Let Q be any other point on the chain. Take the horizontal and vertical through V as axes of coordinates, x and y being the coordinates of Q. Then, for the equilibrium of the bit of chain VQ, the tensions at Q and V must intersect on the axis of x mid-way between V and N, *i.e.* at $\frac{x}{2}$.

Draw the vector polygon for M; then, from the similar triangle,

$$\frac{y}{\frac{x}{2}}=\frac{Wx}{T_0};$$

$$\therefore\ y=\frac{W}{2T_0}\cdot x^2$$

[We might have established this by taking moments about Q when $T_0\cdot y=Wx\cdot\frac{x}{2}$.]

The above construction shews us how to draw a tangent to a parabola (for the tension T at Q is in the direction of the tangent at Q), viz. to draw the tangent at Q, first draw the ordinate QN, bisect VN at M, and join Q to M; then QM is the tangent at Q.

(7) A cable is to be made so that when erected its span will be 50 ft. and dip 15 ft. It is to be of variable section so that the weight per horizontal ft. run is constant and equal to 100 lbs. Draw the tangent at the highest point, and determine the tension at the highest and at the lowest point of the cable.

(8) The span of a suspension bridge is 100 ft., the dip 20 ft., the number of vertical tie-rods for each chain being 8, and the load on each 5 tons; determine the stresses in the links and their lengths.

(9) In Ex. 8 if the tie-rods RU and SV be inclined at 45° to the vertical, determine the stresses in them, and in the supporting pillars PR and QS.

(10) The span being 100, the dip 30, and the number of vertical tie-rods 13 each bearing a load of 4 tons, determine the stresses in them, and the lengths of the links.

(11) Construct the parabola of Ex. 7 and find approximately the ratio of the cross sectional areas at the highest and lowest points. (If O is the lowest and H the highest point, join OH and take any point P_1 on it. From P_1 go horizontally to P_2 on the ordinate at H; mark P where OP_2 cuts the ordinate at P_1. Then P is a point on the parabola. The ratio of a small length of the curve at H to its horizontal projection is approximately the ratio of the cross sections at O and H).

(12) The dip being 5 ft. and the span 100, draw the parabola. The load being 10 lbs. per horizontal foot, find graphically the stresses at the lowest and highest points of the chain. This example is *approximately* the telegraph line problem.

(13) The span being 50 ft. and the load per horizontal foot run being 15 lbs., and the greatest tension allowable being a force of 500 lbs. weight, find the dip, tension and lowest point, and draw the curve assumed by the chain.

Stresses in Frames by Moments.

EXAMPLE. *The frame* QRSTUVW (Fig. 298) *is freely supported at* Q *and* T. *The bars* QV *and* UT *simply cross at* W *and are not pinned there. The loads at* R *and* S *are* 4·8 and 6·2 *tons respectively. If* QT = 30 *ft.*, QV = 22·5 *ft.*, VT = 11·5 *ft. and* SV = 4 *ft.*; *find the stresses in the bars.*

Draw the frame to scale and letter the spaces. Choose a pole P and draw the vector and link polygons. Close the link polygon, and in the vector polygon draw PO parallel to the closing line. In Fig. 298 the reactions at T and Q are CO and OA.

It will be noticed that at all the points $QRSTU$ and V there are three bars meeting (double joints), and that therefore the usual method of resolution is not applicable.

The construction for determining the stress in bg will first be given and then the proof of its correctness.

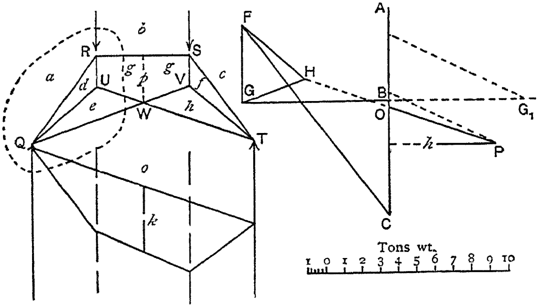


FIG. 298.

Draw a vertical through W cutting the link polygon and RS (bg).

Measure p, the length intercepted on this vertical between W and RS, and k, the intercept cut off by the link polygon.

In the vector polygon draw BG horizontal, and construct (as in Fig. 298) BG, such that $\frac{BG}{k}=\frac{h}{p}$.

BG must be drawn from right to left, not from left to right as BG_1.

Then **BG** is the force exerted by *bg* on *R*, and **GB** is the force on *S*. The stresses in the other bars meeting at *R* or *S* may now be determined, viz. draw *GF* parallel to *gf*, and *CF* parallel to *cf*, intersecting at *F*. The vector polygon for *S* is then *BCFGB*.

For the equilibrium at *V*, draw *FH* parallel to *fh*, and *GH* parallel to *gh*, determining *H*, and the polygon for *V* is *GFH*.

For the equilibrium at *T*, we have *CO*, *OH*, *HF*, *FC*, and since *H* and *O* are already marked we must have *HO* parallel to *ho*. If it is not so, some mistake must have been made in finding *BG*.

Since *QV* and *UT* have two pairs of letters to denote them, *eo* and *gh* and *ge* and *ho*, the stresses in these bars will be given twice over in the stress diagram. See that the results are consistent.

Proof. Suppose the frame cut through as indicated by the dotted curve (Fig. 298). Then any rigid body within may be considered as in equilibrium under the action of the load at *R*, the reaction at *Q*, and forces in the bars *RS*, *UT* and *QV* applied at the cut ends of the bars. The sum of the moments of these forces about any point must be zero; hence, taking moments about *W*:

Moment of force at Q + moment of force at R + moment of force in *RS* must be zero.

The algebraic sum of the first two is given by

$$h\,.\,k,$$

and from Fig. 298 this moment must be negative or clockwise. Hence if s denote the magnitude of the force in *bg* acting on the body

$$hk = p\,.\,s,$$

and the force must push towards *R*, since its moment is positive or contraclockwise.

The bar *bg* or *RS* must therefore be in compression. If, now, we consider the equilibrium at *S*, the force at *S* must push, and therefore, since *BC* is downward, *GB* must be from left to right as indicated.

(14) Find the compression in *bg* by resolving the resultant of the external forces at *Q* and *R* into three along *RS*, *QV* and *UT*.

EXAMPLE. *The frame* PQRST (Fig. 299) *is freely supported at* P *and* Q, *and loaded at* T, S *and* R *with* 1, 2 *and* 1·5 *tons. Find the stresses in the bars.*

$$PT=TS=SR=RQ=9 \text{ and } TPQ=45^\circ$$
$$PU=10{\cdot}6 \text{ and } UPQ=13^\circ,$$
$$SV=6{\cdot}3 \text{ and } VSW=46^\circ.$$

Draw the frame to scale and then determine the reactions at P and Q by the link polygon. It will be seen that the usual process for determining the stresses stops at T and U, since at those points are three bars with unknown stresses in them.

If we make an ideal section, as in Fig. 299, then any rigid body within the dotted curve may be considered as in equilibrium under the vertical forces at T and P and forces in the bars *bj*, *ji*, *io*, and the sum of their moments about S must be zero.

The vertical through S cuts the link polygon in M and N; and hence the sum of the moments of the vertical forces at P and T is given by $MN.P_1X$, where P_1X is the perpendicular from P_1, the pole of the vector polygon, on to the load vector AD.

Also two of the bars *bj* and *ji* pass through S; hence, if s is the stress in *io* and SZ the perpendicular from S on the bar, *oi* the sum of the moments of the stresses in the three bars about S is given by $s.SZ$, and hence

$$s.SZ=MN.P_1X.$$

Construct, then, $s=XY$,

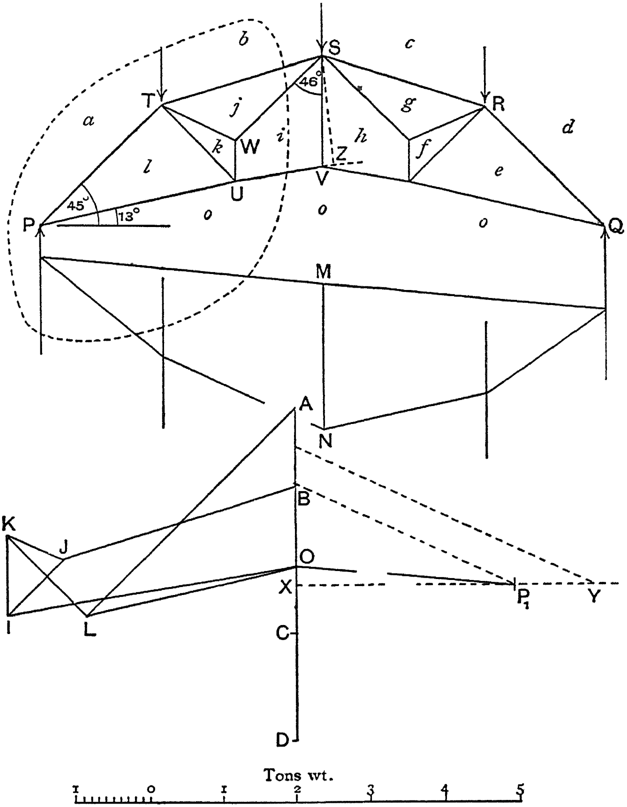
so that $$\frac{XY}{XP_1}=\frac{MN}{SZ},$$

and the stress in *io* is determined.

Again, the sum of the moments of the external forces at T and P is clockwise or negative, hence the moment of T must be contraclockwise, or T must pull on the body enclosed by the dotted line and the bar *io* must be in tension.

Hence set off OI parallel to *oi* and of length XY.

The point L is determined by drawing AL and OL parallel to *al* and *ol*. Hence K is determined by drawing LK and IK parallel to *lk* and *ik*. Since for the point L the action of the bar *lo* is given by *LO* (*OAL* being the vector triangle), for the point *U* we must have the sense *OL*, and since *oi* pulls at U_1, *I* must be on the left of *o*, as in Fig. 299, so that the vector polygon for *U* is *IOLKI*.


a
b
c
d
e
f
g
h
i
j
k
l
o
o
o
S
T
R
W
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U
P
Q
45°
13°
46°
M
A
N
B
K
J
O
X
P₁
Y
I
L
C
D
Tons wt.
1
0
1
2
3
4
5

FIG. 299.

The point *J* is determined by *KJ* and *IJ* parallel to *kj* and *ij*. Hence *B* and *J* are known, and if the drawing has been done correctly, *BJ* will be parallel to *bj*.

Notice that, if a mistake had been made in deciding whether *io* is in tension or compression, the direction of BJ would have been totally different from *bj*; this, then, gives a test as to the correctness of the reasoning and of the drawing.

For the theory of reciprocal figures as applied to stress diagrams the student is referred to Cremona's *Graphical Statics*,* edited by Professor Beare, and to a very elementary work by Professor Henrici and Mr. Turner on *Vectors and Rotors*.†

(15) Fig. 300 gives a not unusual form of truss supporting the roof shelter on a railway platform.

The loads at P, Q, R, S, T, U, V due to the roofing and snow are 0·8, 2·7, 3·8, 5·2, 5, 4·7, 1·5 cwts. Find by moments the stress in WX and compare with that obtained by the usual graphical method. $RT=5$, $PW=3{\cdot}85$, $RW=2{\cdot}3$.

(16) Find the stress in SR, of the cantilever (Fig. 301), by the method of moments. The load at Q is 3 tons. Test the result by resolving the load at Q into three components lying along TV, TQ and SR. $SR=5{\cdot}2$, $RV=5{\cdot}5$, $TV=5{\cdot}25$, $TS=11$, $SV=9{\cdot}4$, $RQ=13{\cdot}5$, $TQ=11{\cdot}4$. Determine the stresses in all the other bars.

(17) As in the previous example, only the load is suspended from a chain which passes over a smooth pulley at Q and is fastened to M, the mid-point of TS.

MISCELLANEOUS EXAMPLES. X.

1. Prove that a chain made of equal links will hang in equilibrium in a vertical plane with the links parallel to lines drawn from a point to equidistant points on a vertical line; and determine graphically the stress at a joint. (Inter Sci., 1902.)

2. A heavy chain is supported by its ends A and B, which are 12 ft. above the lowest part of the chain. The horizonal distance between A and B is 66 ft. and the weight of the chain is 20 lbs. per ft. of its horizontal projection. Draw out to scale (10 ft. to an inch) the shape of the chain and find the force on the chain at the lowest point. What is the maximum force in the chain. (A.M. II., B. of E., 1903.)

3. A suspension bridge two hundred feet span between the centres of the towers has cables having a dip of 30 feet; the backstays are anchored at a distance of 60 feet from the centres of the towers; the load on each cable is 4 tons per foot run. What is the stress on the cables at the centre of the bridge, at the towers, and in the backstays?

(Admiralty Examination, 1904.)

4. Find the stresses in the bars of the trusses shewn, Figs. 302 and 303 for equal loads.

5. Find the stress in ZY of the French roof truss, Fig. 304, by the method of moments and thence the stresses in all the bars. The loads at R, S, T, U, V, W, X being 1500, 2700, 1600, 3400, 1800, 3250, 1750 lbs. weight respectively, $YZ=QZ=ZU=4{\cdot}7$, $QU=8{\cdot}5$, and $QR=14$, and the loads are equidistant.

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* Clarendon Press.

† Published by Mr. Edward Arnold.

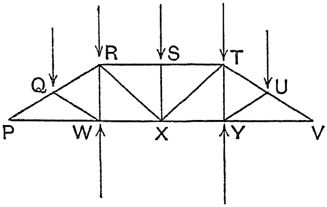

R
S
T
Q
U
P
W
X
Y
V

FIG. 300.

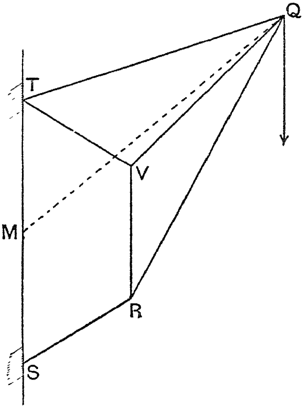
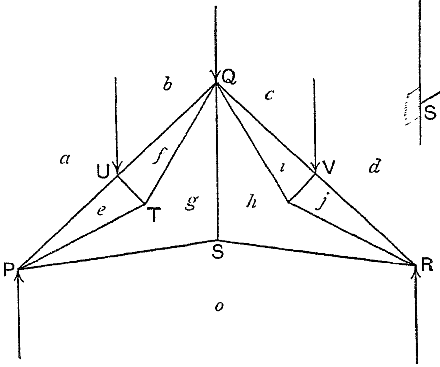


FIG. 301


Q
b
c
a
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ı
V
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e
T
g
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S
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R
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FIG. 302

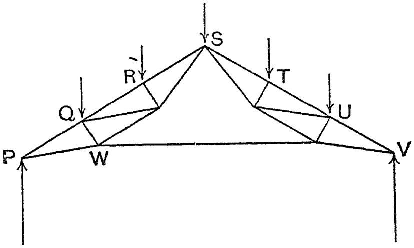

FIG. 303.

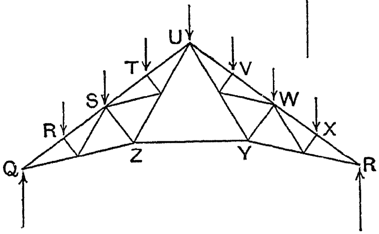
U
T
V
S
W
R
X
Z
Y
Q
R

FIG. 304.

CHAPTER XI.

WORK.

A FORCE acting on a body is said to do work when the body is displaced.

The work done by a constant force acting on a body is defined as the product of the displacement of any point on the axis of the force, and the force component in the direction of the displacement.[*]

Thus, if in consequence of the motion of the body, the point A (Fig. 305) on the axis of the force OF moves from A to A_1, the work done by F is the product $AA_1 . OF_1$, where OF_1 is the force component in the direction of A_1.

If the force component has the same sense as the displacement, work is said to be done *by* the force, and it is considered **positive.**

If the force component has a sense opposite to that of the displacement, work is said to be done *against* the force, and this is considered as **negative work** done **by** the force. The reason for this sign convention is not difficult to see; suppose two forces differing only in sense act on the body, then, so far as motion is concerned, these are equivalent to no force at all, and therefore in any displacement no work is done on the whole. But the components of the forces in the direction of any displacement are equal in magnitude, and opposite in sense, hence the work done by each force must be equal in magnitude, and if one be considered as positive the other must be negative.

---

[*] The other component is supposed perpendicular to the first one.

Unit of Work. In Statics the unit of work is usually taken as the **foot-pound,** or the work done by a force of 1 lb. weight when the body is moved 1 ft. in the direction of the force.

If the unit of force be a **dyne,** the unit of work is called an **erg,** and is the work done by one dyne when the body is displaced 1 centimetre in the direction of the force.

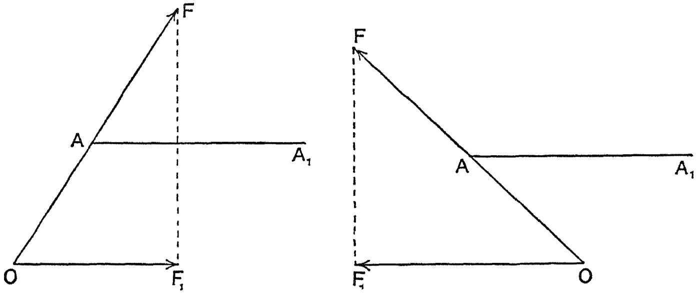

FIG. 305.

Graphical Representation. Work done is represented graphically by the area of a rectangle of which one side represents to scale the displacement, and the adjacent side the force component in the direction of the displacement. To measure this area the rectangle is reduced to unit base either (i) the unit of length, when the altitude is measured on the force scale, or (ii) the unit of force, when the altitude is measured on the length scale (pp. 38-40).

Moment of a Force and Work done by a Force. These are both represented graphically by an area, but have totally different physical interpretations. The moment of a force is a vector quantity, its plane is determined by the plane of the force and the point, and its sense by the sense of the force. The area representing the work done by a force has no special plane, and may be supposed anywhere; it is a scalar area though it may be positive or negative.

EXAMPLE. *The shafts of a carriage are inclined at an angle of* 15° *to the horizontal. If the pull transmitted along the shafts from the horse be* 123 *lbs. weight, find the work done by this force in moving the carriage through* 137 *ft.*

Draw vertically upwards a line OD (Fig. 306), of length 6·85″, and set up along it $OU = 5''$. Through O draw (i) a horizontal line OF_1, and (ii) OF sloping at 15° and of length 12·3 cm. Draw FF_1 perpendicular to OF_1, and mark on OF_1 the point W, where DW, parallel to UF_1, cuts it.

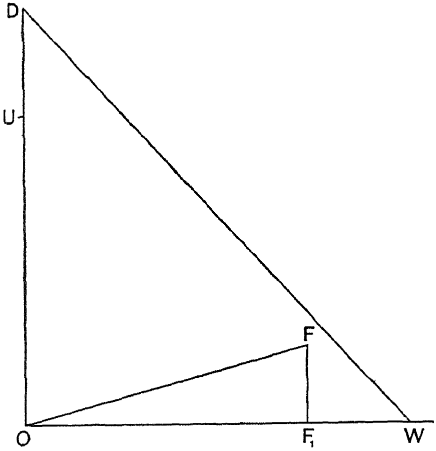

FIG. 306.

Measure OW on the force scale, and multiply by 10; this gives the number of foot-lbs. in the work done by the horse on the carriage.

Proof. OF_1 is the component of OF in the direction of the displacement, and hence the work done is measured by $OF_1 . OD$, *i.e.* by the area of the rectangle having OF_1 and OD as adjacent sides.

This rectangle is equal in area to $OU . OW$, and OU represents 10 feet; hence measuring OW on the force scale gives the work done in 10 ft.-lbs., and therefore $10 \times OW$ on the force scale gives the work done in ft.-lbs.

(1) The weight of a bucket of water is given by a line of length 47 cm. (scale 1″ to 10 lbs.). Find the work done in ft.-lbs. against gravity in raising the bucket from the bottom to the top of a well $76\frac{1}{2}$ ft. deep.

(2) A horse pulls a canal boat with a force of 151 lbs. weight, the tow rope makes an angle of 25° with the bank. Find the work done in ft.-lbs. on the boat in pulling it along 117 ft.

(3) If a hole is punched through a metal plate 0·78 inches thick, and the average resistance to the force of the punch is of magnitude 23700 lbs. weight, find the work done in ft.-lbs.

(4) A weight of 1720 lbs. by falling through 27·8 ft. lifts, by means of a machine, a weight of 970 lbs. through 47·3 ft. Find the total work done by gravity.

(5) The inclination of a plane is 25°; find the work done against gravity in pushing a body weighing 7·3 cwts., 15·7 ft. up the plane.

(6) If the body be pushed up the plane by a horizontal force of 8·2 cwts., find the work done by this force.

The work done by a force when a point in its axis is displaced, is the product of the force and the component displacement in the direction of the force.

This is really an alternative definition to that given on p. 354.

Let OD (Fig. 307) represent the displacement, and OF the force under consideration; then, according to the first definition, the work done $= OF_1 \,.\, OD$, where FF_1 is perpendicular to OD.

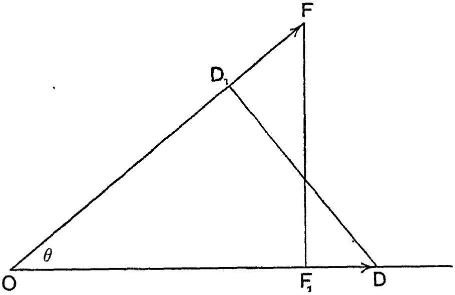

FIG. 307.

According to the alternative definition the work done $= OF \,.\, OD_1$, where DD_1 is perpendicular to OD_1.

But OF_1F and OD_1D are similar triangles,

and hence $$\frac{OF_1}{OF} = \frac{OD_1}{OD} \text{ or } OD \,.\, OF_1 = OD_1 \,.\, OF.$$

More shortly, if θ is the angle between the force and the displacement, and f and d are their magnitudes, then the

$$\text{work done} = f \,.\, d \cos\theta \text{ (according to second definition)}$$
$$= f\cos\theta \,.\, d \text{ (according to first definition).}$$

Work and Motion. It should be observed that for a force to do work or work to be done against a force, motion is essential. Unless some point on the line of action of a force moves, and the displacement has a component in the direction of the force, no work is done by or against the force. Thus, however great the force which a horse exerts on a cart in trying to start it, no work is done by this force *on the cart* unless the cart moves. If by means of a second horse the cart be made to move, then the first horse does work on the cart, the amount being his pull multiplied by the component displacement. If a force acts on a body at right angles to its displacement, no work is done by the force; thus in the case of a body pushed along the surface of a horizontal table no work is done by the weight of the body because its line of action is perpendicular to the displacement.

If, then, we know the work done by a force to be zero, we may have either (i) no displacement, or (ii) a displacement perpendicular to the force.

We are not here directly concerned with the force or forces to which the motion as such may be due. For instance in Exercise 1, the actual work done by the person raising the bucket is not the same as the work done against gravity. To find the former from the latter we must know the speed of the bucket and the work spent in giving it kinetic energy as well as the work done against the resistances.

It is true, of course, that if we knew the force exerted by the man the work done would be this force × the displacement. The difference between this work, and the work done against gravity, gives the energy imparted to the bucket, and the work done in overcoming resistances. Similarly, in the example on p. 356, we are concerned with the work done by the force applied to the carriage; this may not be the same as the work done against the resistances to the carriage motion, because the carriage may be going faster at one time than at another.

Shortly put, we are concerned in Statics only incidentally with the forces causing motion; our problem always is to find the work done by certain given forces when the body is displaced, the work being measured according to the definition on p. 354.

Displacement and Actual Path. The actual path of the displaced point is immaterial; so long as the displacement is the same, the work done will be the same.

Let F be the force and $\mathbf{AA_1}$ the displacement of A, then (Fig. 308) the work done is $F.AA_2$.

If the displacement had been first from A to B, and then from B to A_1 the work done would have been

$$F.AB_2+FB_2A_2=F.AA_2.$$

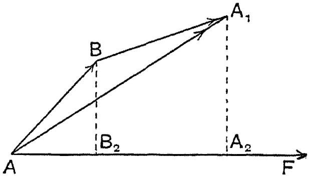

FIG. 308.

This decomposition of displacements may be supposed repeated without limit, so that A may be supposed to move on any curved path from A to A_1, and the work done by F will still be $F.AA_2$.

(7) Draw a circle of 3″ radius, and suppose it to represent a vertical wheel of radius 6′. Find the work done by gravity when a load of 0·34 ton is moved round the wheel from the lowest position through one, two, three and four quadrants respectively.

Change of Direction of Force. If the force changes its direction as the point on its axis moves, but the angle between the force and the direction of the motion remains unaltered, the work done will be the product of the distance moved through by the point and the force component in the direction of the motion at any instant.

Thus, if the point move in a circle, and the force is always a tangent to the circle, the work done in a complete revolution will be the force × the length of the circumference.

(8) A body is moved through a circular arc, of length 25 ft. and radius 19 ft., by a force of 34 lbs. weight, which always makes an angle of 70° with the radius. Find the work done on the body in ft.-lbs.

(9) A man pushes at a capstan bar with both hands. One hand, at a distance of 9 ft. from the axis, pushes perpendicular to the bar with a force of 30 lbs. weight, the other pushes with a force of 38 lbs. weight at a distance of 7·8 ft. from the axis, and inclined to the bar at an angle of 72°. Find the work done in ft.-lbs. during a complete revolution.

Work done against Friction. In the Chapter on Friction it was explained that the coefficient of friction was the ratio of the force tending to produce motion to the normal pressure when the body was just on the point of motion. Such a coefficient of friction is therefore not at once applicable to bodies in motion without further *experimental evidence.*

Experimental Laws of Friction for Bodies in Motion. It has been found for bodies actually sliding one on the other that the friction between them is

(i) proportional to the normal pressure;
(ii) independent of the relative speeds of the bodies;
(iii) ,, ,, area in contact;
(iv) dependent on the nature of the surfaces;

so that for bodies in motion, if F denoted the friction and N the normal pressure, $F = \mu N$,
where μ is constant for any two particular surfaces, but varies for different surfaces and is called the coefficient of dynamical friction. This coefficient of dynamical friction is slightly less than for limiting friction.

EXAMPLE. *A rough plane is of length* 13 *ft. and height* 7·8 *ft. Find the work done by the least possible equilibrating force* when the body of weight* 24·8 *lbs. is displaced from the bottom to the top of the plane. The angle of friction for the plane and body is* 18°.

Draw the plane ACB (Fig. 309) to scale ($AC = 13$ cm. say), and draw ON perpendicular to AC.

---

* A body is not necessarily at rest when in equilibrium; it may be moving with constant velocity.

Set off OW vertically to represent the load of 24·8 lbs. weight, and draw OP, making 18° with ON, on the side away from OW, and then WP perpendicular to OP.

Project WP to W_1P_1 on the plane by lines parallel to ON. Along W_1W set off

$W_1Q = 13$ cm.

and $W_1U = 10$ cm.

Draw QQ_1 parallel to UP_1. Measure W_1Q_1 on the force scale, and multiply by 10; it gives the number of ft.-lbs. of work done in sliding the body up the plane.

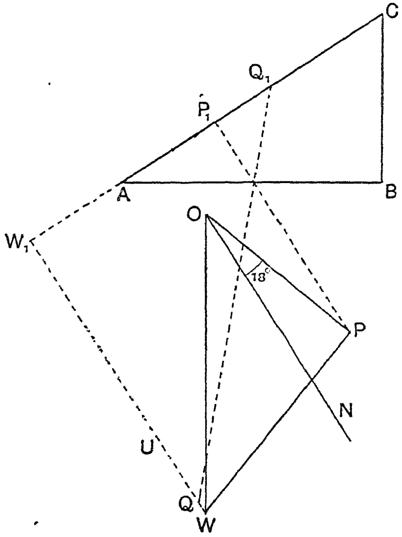

Fig. 300.

(10) Find the work done against gravity and also that done against friction. What connection is there between these and the work done by the equilibrating force?

(11) Find the work done when the body is displaced up the plane by the equilibrant when (i) parallel to the plane, (ii) horizontal.
Find also the work done against friction.

(12) A man pushes a roller up a hill rising 1 in 7, and keeps the handle horizontal. The resistance is equivalent to a force of 24 lbs. acting down the hill. Find the work done on the roller by the man in moving it 12·3′ up the hill, if the roller weighs 268 lbs.

(13) The axle of a fly-wheel has a radius of 2″, the weight of the wheel is 1780 lbs. and the coefficient of friction for the axle and bearing is 0·18. Find the work done in ft.-lbs. against the friction per revolution of the wheel.

If a given set of forces acting on a body would keep it in equilibrium, then the total work done by all the forces is zero during any displacement of the body, the forces being supposed constant.

This is a direct consequence of the fact that the vectors of the forces form a closed polygon, and therefore the sum of their components in any direction is zero.

In many cases the forces cannot be supposed constant for finite displacements, and we have to consider infinitely small displacements. In the case of a beam leaning against a smooth wall and kept from sliding down by a peg at its foot, the reactions depend on the slope of the beam, and we cannot at one and the same time suppose the slope altered and the reactions unaltered.

M.C. and Work Done. **The work done against gravity in raising a body of weight W is equal to the work done in raising a mass of weight W supposed concentrated at the mass-centre of the body.**

Let w_1, w_2, w_3, ... be the weight of the particles of the bodies, y_1, y_2, y_3, ... their initial vertical distances above some horizontal plane, and Y_1, Y_2, ... their final distances above the same plane. Then the work done against gravity is

$$w_1(Y_1 - y_1) + w_2(Y_2 - y_2) + \ldots = \Sigma w_1 Y_1 - \Sigma w_1 y_1.$$

But $\Sigma w_1 y_1 = \bar{y}\Sigma w_1$ where $\bar{y}$ is the initial vertical distance of the M.C. above the plane

and $\Sigma w_1 Y_1 = \bar{Y}\Sigma w_1$ where $\bar{Y}$ is the final vertical distance.

Therefore, total work done against gravity $= (\bar{Y} - \bar{y})W$ the work done on a particle of weight W in lifting it through the distances $\bar{Y} - \bar{y}$.

(On each particle of the body other forces than the weight act, viz. the pushes and pulls of adjacent particles. These pushes and pulls constitute the stress of the body, and since the stress consists of equal and opposite pairs of forces, the total work done by these is zero.)

(14) Find by calculation the work done in emptying a cylindrical well shaft of diameter 3 ft., the depth of the well being 110 ft. and the top of the water being 26 ft. below the surface; the weight of a cubic foot of water is 62·5 lbs. approximately.

(15) ABC is a triangular prism weighing 78·3 lbs. It rests on the ground with the face BC, of length 2·37 ft., in contact with it. BA is vertical and of length 3·16 ft. Find the work done against gravity in turning the prism so that it is about to fall over (i) round the edge at B, (ii) round the edge at C.

(16) Find by calculation the work done against gravity in raising a cage of weight 727 lbs. from a depth of 236 ft. by a wire rope of that length, the rope weighs 5·7 lbs. per yard.

Work done by a Variable Force.

EXAMPLE. *A force moves a body in its line of action; for successive displacements of* 1 *ft. the magnitude of the force is* 51·3, 72·4, 65·7, 42·6, 31·5, 27·1, 30·3, 39·2, 46·9 *lbs. weight. Represent the work done graphically and find its amount in ft.-lbs.*

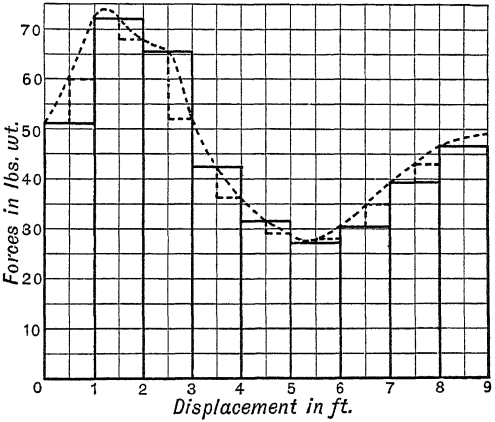

FIG. 310.

On squared paper take two axes of co-ordinates, the horizontal one to represent displacements to the scale of 1″ to a foot and the vertical one to represent forces to the scale of 0·1″ to a lb. weight.

Plot the points corresponding to the numbers, and complete the rectangles as indicated in Fig. 310. Evidently the whole area of the figure represents the work done. Find the area in sq. inches; the number of sq. inches multiplied by 10 gives the work done in ft.-lbs.

Suppose we knew that the force changed for every displacement of 6″, being 60, 68, 52, 36, 29, 28, 35, 43 and 47 lbs. weight at the intermediate points. Then, plotting these points, we see that the total work done is represented by the area of the new figure.

Further, if the force changed continuously instead of suddenly, we should have, instead of a succession of points forming a zig-zag line, a continuous smooth curve, and we see that the area enclosed by this curve, the axes of coordinates and the ordinate at 9 represents the work done.

(17) The resistance to the motion of a car for various displacements is given in the accompanying table. Draw a curve giving the relation between the displacement and the resistance. Divide the total displacement into ten equal parts and erect ordinates at the mid-points. Add the mid-ordinates by means of a straight strip and measure on the force scale; multiply by the width of the strips in feet. The product is the work done against the resistance in ft.-lbs.

Displacement in ft.	10	30	50	70	90	110	130	150	170	190
Resistance in lbs. weight	160	184	231	289	394	540	641	709	751	776

(18) The force in lbs. weight acting on a body is always twice the magnitude of the displacement in feet and acts in the direction of the displacement. Find the work done by the force for a total displacement of 17·4 feet.

Work done in Spring Extension or Compression. Experiments shew that when a helical spring is extended beyond its natural length, the extension is always proportional to the force applied (up to the elastic limit of the spring), so that if e denote the extension, and W the load, $\frac{W}{e}$ is, for the same spring, always constant (Hooke's Law).

If the results of an experiment be plotted, say extension in cms. horizontally, and load in grms. weight vertically, the points will be found to lie approximately in a straight line—as in Fig. 311—passing through the origin of coordinates. From the graph we see that $\frac{W}{e}$ is always constant and is measured by $\tan \alpha$.

The work done in extending the spring a distance OA is therefore given by the area OAB to scale, and is measured by $\frac{1}{2}OA\,.\,AB$, *i.e.* $\frac{1}{2}$ the product of the maximum extension and maximum load.

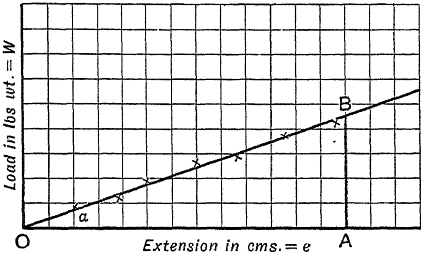

FIG. 311.

(19) A spring is found to extend a distance of 12·7 cms. beyond its natural length under a load of 46·3 lbs. Find the work done in inch-lbs. in gradually extending the spring from its natural length to 8·7 cms. beyond.

(Draw the straight line graph by setting off $OA = 12{\cdot}7$ cms. horizontally, and AB vertically a distance of 4·63 inches, and join OB; OB is the graph. Find the area enclosed by OB, OA, and the ordinate at a point 8·7 from O.)

(20) A spring is found to extend a distance of 15 cms. under a load of 17·6 lbs. Find the work done in gradually extending it from an extension of 7·3 cms. to one of 17·4 in inch-lbs.

(21) A bar is fixed at one end, and twisted by means of an arm of length 1 ft., fixed at right angles to the length, at the other end. To keep the free end of the bar twisted through a radian requires a force of 27 lbs. weight to be applied to the end of the arm. The force applied being proportional to the angle of twist, find the work done in twisting the free end from 0·56 to 1·32 radians.

Work done in compressing Gases. For many gases at ordinary temperatures and pressures the relationship between the volume they occupy and the pressure to which they are subjected is given by the law

$$\text{Pressure} \times \text{volume} = C \text{ (where } C \text{ is constant)}$$

(known as Boyle's or Marriotte's Law).

EXAMPLE. *A gas obeying Boyle's Law is enclosed in a cylinder fitted with an air-tight piston; to find the work done in compressing the gas.*

Suppose when the compression starts at the atmospheric pressure of 14·3 lbs. per square inch, the volume of the enclosed gas is 100 cubic inches, then the constant above is $14 \cdot 3 \times 100 = 1430$. Suppose the volume is reduced to 20 cubic inches.

On squared paper take two axes of coordinates, the vertical one for pressures, and the horizontal one for volumes. Along the latter set off OM to represent to scale 100 cubic inches, and set up vertically MP to represent the pressure of the gas (14·3 lbs. per sq. inch).

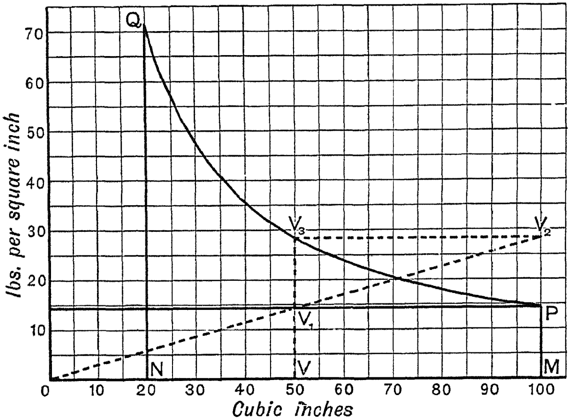

FIG. 312.

Draw a horizontal line through P (Fig. 312), and mark any point V_1 on it; put a straight edge along OV_1, and mark the point V_2 where it cuts the ordinate PM. From V_2 go horizontally to V_3 on the ordinate VV_1 through V_1.

V_3 is a point on the curve whose equation is

$$P \,.\, V = 14 \cdot 3 \times 100 = OM \,.\, MP.$$

Repeat the construction for a number of points like V_1, and obtain, say, nine points between $V = 20$, and $V = 100$. Join the points by a smooth curve QV_3P. This curve has for its equation

$$P \,.\, V = 14 \cdot 3 \times 100.$$

Find the area enclosed between the curve, the axis OM, and the ordinates QN and PM, by the mid-ordinate method, *i.e.* add the mid-ordinates of a number of equally wide strips between N and M by the strip method, and multiply by the common width of the strips, the former being measured on the pressure scale, and the latter on the volume scale.

The product is the work done in compressing the gas in inch-pounds.

Proof. Pressure meaning force per sq. inch, then if A is the area of the piston in square inches, the total force acting when the pressure is P is $P \,.\, A = F$, say.

If L is the length of the cylinder in inches, then

$$V = AL,$$

and

$$PV = \frac{F}{A} \,.\, A \,.\, L = F \,.\, L,$$

and we may take the ordinate to represent the force on the piston, and the abscissae to represent displacement.

(22) Find the work done in compressing a gas from a volume of 7·32 to 3·64 cu.-ft., if the initial pressure of the gas was 5200 lbs. weight per sq. ft.

*(23) The resistance to the motion of a body in a liquid varies as the square of the speed, find the work done in reducing the speed from 20 to 11·5 miles per hour in a distance of 1·6 miles, if the resistance to the motion is a force of 12·3 tons weight, when the speed is 8·6 miles per hour.

*(24) If W_0 is the weight of a body on the earth's surface, and W the weight of the same body at a distance R from the earth's centre, then $WR^2 = W_0R_0^2$ where R_0 is the radius of the earth. Find the work done on a meteorite by the earth's attraction in moving it from $R = 8000$ miles, to $R = R_0 = 4000$ miles, if the meteorite weighs $\frac{1}{2}$ a ton on the earth's surface.

MISCELLANEOUS EXAMPLES. XI.

1. Explain the term work.

Find the work done in raising a lift full of people and weighing 2 tons through a height of 80 feet. Explain the system of units you use.

(Engineer Students, 1903.)

2. A uniform rope hangs by one end, and carries a weight at the other; shew how to draw a diagram to represent the work done in winding up the rope, and thereby lifting the weight. (B. of E., II.)

3. Draw a line Ax, and take in it five equidistant points, B, C, D, E, F; suppose a force (P) to act along Ax, and that its value at the points B, C, D, E, F are respectively 50, 35, 28, 25, 24 lbs., let the distances

BC, *CD*, ... represent 3 ft. apiece; draw a diagram of the work done by the force, and calculate (by Simpson's rule, if you know it) in foot-pounds the work done by the force while acting from *B* to *F*. (B. of E., II.)

4. Draw a diagram of work in the following case: Six equal weights (W) are fastened to a rope in such a way that one follows another at distances of a foot. The rope hangs vertically with the lowest weight 3 ft. above the ground; if the rope be gradually lowered draw a diagram for the work done by gravity on the bodies, when all have come to the ground. (B. of E., II., 1903.)

5. A load of 10 cwts. is raised from the bottom of a shaft 500 feet deep by a wire rope weighing 2 lbs. per foot. The rope is wound up on a drum, 3 feet in diameter. Draw a curve, showing the moment exerted at the drum throughout the motion, and find the whole work done during the lift. (Patent Office, 1905.)

6. Shew how to represent in a diagram the work done by a force P of variable magnitude, which displaces its point of application in its own fixed line of action from A to B. Let P begin with the magnitude 50 lbs. weight, and keeping its magnitude constant, displace its point of application from A to C, a distance of 2 feet; then from C to D, a distance of 8 ft., let P vary inversely (without discontinuity) as the distance of its point of application from A. Draw the work diagram and find the total work done from A to B. (Inter. B.Sc. (Engineering), 1906.)

7. Give the vector definition of the mass-centre of a system of particles. Prove that the work done against gravity in moving a system from one configuration to another is equal to the work done in lifting a particle equal to the total mass from the first position of the M.C. to the second. (Inter. B.Sc. (Engineering), 1905.)

8. The table below gives the relation between pressure and volume for 1 lb. of saturated steam, between certain limits of pressure. Plot a graph which will show this relation, and by counting squares on the sectional paper, determine the area bounded by the curve, the horizontal axis or line of zero pressure, and the limiting ordinates (parallel to the line of zero volume).

If for any small change of volume of the steam the product of pressure in lbs. per square foot and the change of volume in cubic feet represents the work done in foot-lbs., find how many foot-lbs. of work will be done in compressing the steam from a volume of 4·29 c. ft. to a volume of 1·53 c. ft.

Pressure in lbs. per square inch.	Volume in cubic feet.
101·9	4·29
115·1	3·82
129·8	3·42
145·8	3·07
163·3	2·76
182·4	2·48
203·3	2·24
225·9	2·03
250·3	1·84
276·9	1·68
305·5	1·53

(Military Entrance, 1905.)

APPENDIX.

EXPERIMENTS ON MOMENTS.

The Lever. A good simple lever can be made from a metre scale. Leaving about 3 cms. untouched at the centre, cut away from one edge down to the middle line and remove the end portions as in Fig. 313. Bore a hole just above the middle line at the mid-point of the length, and fix a steel cylindrical peg firmly in the hole so as to protrude about ½″ on both sides.

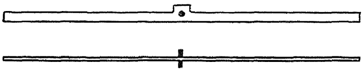

FIG. 313.

The lever should be supported between two wooden blocks of the same height or on a special stand. Weights may be suspended from the lever by looped threads, either directly or by scale pans.

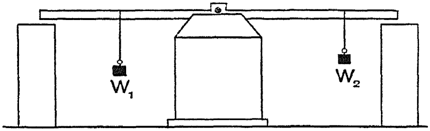

FIG. 314.

Stops should be provided to prevent the lever overbalancing when the weights are not properly adjusted.

If the lever is not horizontal when placed on its supports, cut away a little more wood from the heavier side until it is.

EXPT. I. Suspend, by means of a silk loop, a 100 gramme weight from the left-hand side of the lever at 20 cms. from the fulcrum. From the right-hand side suspend a weight of 40 grammes and adjust its position until the lever is horizontal.

Do the same with weights of 50, 60, 100, 150, 200, 250, 300, 400 and 500 grammes on the right-hand side, noting in each case the distance of the supporting thread from the fulcrum.

Tabulate the results thus :

Weight in grammes = W,	40	50	60	100	etc.
Distance from fulcrum in cms. = d, - - -					

On squared paper take a horizontal line to represent distances from the fulcrum, scale 1 in. to 5 cms., and vertical distances to represent the weights, scale 1 inch to 50 grammes, and mark the divisions as indicated in the figure.

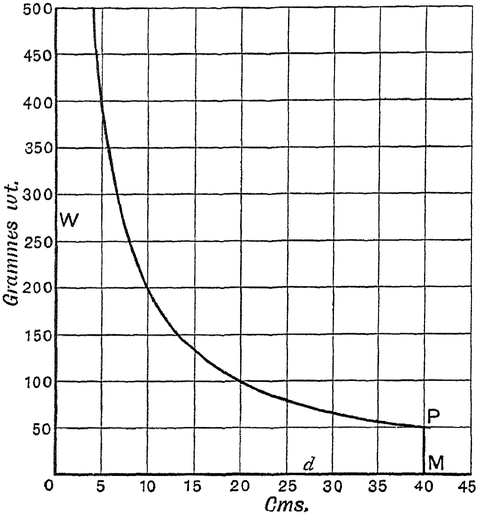

FIG. 315.

Plot the points whose coordinates are given in your table and draw a smooth curve lying as evenly as possible amongst them.

The curve can be recognised as like the one constructed on p. 34. Verify this by multiplying the weights by the distances (take the weights from the curve).

Find that always Weight × by distance from fulcrum is the same.

See if this constant product is also the product of the left-hand weight and its distance.

These products are called the *moments* or turning moments of the weights about the fulcrum, and we have *moment of weight on one side = moment of weight on the other.*

From the graph determine

(i) At what distance from the fulcrum a weight of 120 grammes on the right would balance the left-hand weight?

(ii) What weight must be placed at 27·5 cms. on the right to balance the left-hand weight?

(iii) Suspend a $\frac{1}{2}$ lb. weight from the right arm and determine by trial the point at which it is in equilibrium. Read off from the curve its weight in grammes.

(iv) If all the weights used on the right in the experiment were suspended at the same time from their old positions, what weight would have to be suspended at 30 cms. from fulcrum on the left?

(v) Where would a 500 gramme weight have to be suspended on the left to balance all the weights used on the right?

EXPT. II. Generalise Expt. I. by taking three weights on one side and two on the other at the same time.

If M_R and M_L mean the moment of a weight on the right and left respectively, then the result of this experiment may be symbolised by the equation

$$\Sigma M_L - \Sigma M_R = 0.$$

Notice that the weights on the left tend to produce a contraclockwise rotation of the lever, and those on the right a clockwise, hence if the former be reckoned positive and the latter negative, then, as an algebraic sum, *the sum of all the moments about the fulcrum is zero, or* $\Sigma M = 0$.

Repeat this experiment twice, using different weights, and see if $\Sigma M = 0$ in these cases.

(1) Weights 20, 25 and 100 grammes are placed at distances 45, 30 and 20 cms. to the right of the fulcrum. A weight of 70 grammes is placed at 10 cms. on the left; where must 20 grammes be placed to balance the others, and what weight must be placed at 40 cms. to the left to balance the same set?

(2) Would an upward push of 200 grammes at 15 cms. on the right be likely to have the same balancing power as 200 grammes hanging at 15 cms. on the left? (A 200 grammes upward push at any point would be counter-balanced by 200 grammes weight hanging vertically below.)

(3) *AB* represents a lever 25 cms. long, the fulcrum being somewhere between *A* and *B*. A weight of 170 grammes at *A* balances 60 at *B*; determine graphically the position of the fulcrum.

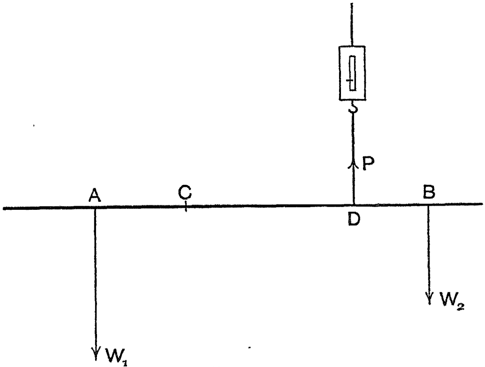

FIG. 316.

EXPT. III. Generalise the results of Expt. II. by applying an upward force on the right side by means of a spring balance (Fig. 316).

Shew in several cases that

$$W_1 . AC = W_2 \times CB - P . CD \text{ (where } C \text{ is the fulcrum),}$$

P being the upward pull of the spring as registered on the scale.

(4)						
	Forces in lbs. weight,	10	−15	25	−12	20
	Distances from it in inches,	2	3	−10	7	

The above table gives the parallel forces acting on a lever, and the distances of their points of application from fulcrum. Distances to the right are given as positive, those to the left as negative; a −force means an upward push. Find the distance of the 20 lbs. weight from the fulcrum for equilibrium. On which side must it be placed?

(5) Expt. I. Chap. IV. shewed that a force acting on a rigid body may be supposed applied at any point in its line of action. Suppose, then, a weight on one side of a lever were suspended by a thread cemented on to the lower edge (instead of strung from the upper edge), would this make any difference in its balancing power? If not, the moment would in this case be expressed by weight × ?

Expt. IV. A farther generalisation of Expt. III. can be made by changing the shape of the lever.

Fix a cylindrical peg through the centre of a rectangular piece of planed wood (say 12″ × 10″ × ½″). Fix two drawing pins firmly near opposite edges of the board, leaving sufficient space between the heads of the pins and the board to insert a loop of thread. Balance the board by means of a peg on two blocks as in the case of the metre scale lever. Hang weights of 400 and 500 grammes on the pins, and let the lever take up a position of equilibrium. Mark the lines of the strings on the board. Measure the perpendicular distances of the strings from the fulcrum. See if the moment law, Σ(weight × perpendicular distance from fulcrum) = 0, is true.

Remove the pins and mark their old positions; set the pins in other positions on the two lines drawn on the board, and suspend the weights as before. Is the lever in equilibrium in the same position as before? How can you tell that the position is the same?

How does the experiment verify the deduction from Expt. III. Chap. IV.

Notice that when the line joining the two pins is above the fulcrum the equilibrium is unstable.

Expt. V. A final generalisation is effected by arranging an experiment in which the forces have different directions.

Cut out a piece of irregular shaped cardboard. Punch five holes in it, and fix it on the vertical drawing board (Expt. II. Chap. IV.) by means of two stout pins. Attach hooks, threads and weights as in Expt. IV. of Chap. IV., one weight hanging vertically. Remove one of the pins; the card will probably turn round the remaining pin, and take up some position of equilibrium. The card is now a lever in equilibrium, the pin being the fulcrum.

Draw lines on the card shewing the axes of the forces, and indicate on the lines the magnitudes and senses of these forces.

Dismount the card. Measure the perpendiculars from the fulcrum on the axes of the forces, and calculate the sum of the moments.

Find the sum of the vectors of the forces.

Draw through the fulcrum a line parallel to the sum of the vectors. Mark a point on this line, and mark also a point not near the line.

Calculate the sum of the moments about these two points as if they were fulcra.

If a force equal in magnitude and direction, but opposite in sense, to the resultant vector acted in the axis drawn through the fulcrum, what would be the sum of the moments of the whole six forces about these three points, viz. the original fulcrum, and the two marked points?

What was the force acting at the fulcrum on the card, *i.e.* the reaction of the pin?

Expt. VI. Fix the same card to the drawing board (by two pins) in such a position that the axis originally vertical is vertical again. Arrange the pulleys so that the same forces may act on the card as in the last experiment, and in addition a sixth force given by the reversed resultant vector whose line of axis has been already drawn.

Remove the pins and see if the card is in equilibrium.

If in equilibrium any point on the card pinned to the board would do for a fulcrum. Calculate the sum of the moments of the six forces about several points.

Was the vector polygon closed for all the forces?

Would there have been equilibrium if the sixth force had been applied in any other line than the one through the fulcrum?

Deductions.

(1) If a rigid body is free to turn about an axis (fulcrum) the rotative tendency of any force acting on the body is measured by the product of the force and the perpendicular on its axis from the fulcrum, and this moment is positive or negative according as the rotative tendency is contra-clockwise or clockwise. Expts. I.-VI.

(2) If a number of coplanar forces act on a body free to turn about an axis, the body will turn until the sum of the moments of the forces about the axis is zero. If the body, free to turn about an axis, does not, the sum of the moments of the forces about that axis is zero. Expt. V.

(3) If the sum of the moments of the forces is zero about three points (non-collinear) in the plane, the body will be in equilibrium (Expts. V. and VI.), and if the body is in equilibrium, the sum of the moments is zero for all points. Expt. VI.

NOTE A.

CIRCULAR MEASURE OF AN ANGLE.

DRAW a circle of radius $3''$ and mark by radial lines angles of 30°, 60° and 90°. Step off the arcs, corresponding to these, along a tangent and measure their lengths in inches. Divide these lengths by the radius. The numbers so obtained are approximately the circular measures of the angles. See that the last number is 1·57 and that the numbers are in the ratio $1:2:3$, or nearly so.

Whatever the radius of the circle the circular measure of these angles will be the same.

If an arc be stepped off equal to the radius, the angle at the centre will have unity as its circular measure. This angle is called a **radian** and is the unit of circular measure. Construct a radian and see that it is nearly 57·3°.

The measure of an angle in radians is always $\frac{\text{arc}}{\text{radius}}$, which being the ratio of two lengths is a number.

Mathematicians formerly went to enormous trouble to calculate the measure of two right angles in radians; it has been worked out to 707 decimal places. The number of radians in two right angles is always denoted by the letter π. Correct to 5 figures its value is 3·1416; the fraction $\frac{22}{7}$ gives π correct to three figures. Instead of giving the radians in an angle as a number, approximately correct, it is sometimes convenient to express it as a fraction of π; thus the number of radians in 15°, 30° and 45° are $\frac{\pi}{12}$, $\frac{\pi}{6}$ and $\frac{\pi}{4}$.

Since $\pi=\frac{\text{semicircular arc}}{\text{radius}}$, the circumference of a circle of radius r is $2\pi r$.

NOTE B.

GREEK LETTERS USED IN THE TEXT, WITH THEIR USUAL PRONUNCIATIONS.

α (Alpha).
β (Beta).
γ (Gamma).
δ (Delta).
ϵ (Epsilon).
θ (Theta).
π (Pi).
ρ (Rho).
$\left.\begin{matrix}\sigma\\ \Sigma\end{matrix}\right\}$ (Sigma).
ϕ (Phi).

ANSWERS.

CHAPTER I.

Exercises. Pages 1–41.

3. 0·62″; 38·4′. **5.** Total load, 15·1 ozs. **6.** 11·3 ozs.

7. 123. **8.** 27·4. **9.** 12. **10.** −9·7.

15. Complete the triangle on a, and measure the angle between u and the third side. Through the ends of b draw lines making θ and 60° with b, then c is the side bounding the 60° angle.

16. 8·88. **25.** 1·92. **29.** 1·36, 1·48, 0·57. **32.** 1·47, 0·83. **34.** 9·15.

35. 6·38. **36.** 6·06. **38.** 0·575. **39.** 2·88. **40.** 225.

41. 1·28, 1·64, 2·08, 2·67, 3·43, 4·39. **42.** 0·73, 0·62, 0·53.

43. 1·13, 1·06, 0·92, 0·96.

46. 0·04, 0·13, 0·21, 0·82, 0·97, 1·74, 1·90.

47. 0·63, 0·87, 0·96, 1·17, 1·30, 1·43, 1·48.

50. $y=\frac{x^2}{25u}$, where u is 1″; $y=x^2$; 9″.

51. $\frac{y}{b}=\frac{x^2}{a^2}$; 1·5″ and 13·5″ and are independent of a.

64. 3·8. **65.** 8130. **66.** 10·9″. **67.** 257; 694.

69. 31·2 million ergs. **70.** 10·4 secs. **71.** 2·94 ft.-lbs. **72.** 17·4 ft.-tons.

73. A line of length 8·96 cms., scale 1 cm. to 1 lb.-ft.; a line of length 8·96″, scale 1″ to 1 lb.-ft.

Miscellaneous Examples I. Pages 41, 42.

1. 0·6″, 1·42″, 0·7″. **2.** 5·14 cms. **4.** 1·88″.

5. 7·85, 12·6, 15·7. **6.** 45·3, 35·0, 1·7. **7.** 1·07, 1·38, 2·36, 2·84.

8. $y=\frac{x^3}{16u^2}$, where u is an inch. **9.** 2·47. **10.** 21 ft.-lbs.

11. 2·46, 1·19, 0·76, 4·0. **13.** 0·31 cub. ft.

CHAPTER II.

Exercises. PAGES 43–64.

1. 5·12 sq. ins.
2. 20·4 sq. cms.
4. 2·28 sq. ins.
6. 9·1 sq. ins.
7. 2420 sq. ft.
8. 1·67 sq. ins.
9. *ABCD* is − 1·03 sq. ins.
11. 9 sq. ins.
12. 4950 sq. yds.
13. 8·87.
16. 5·9 sq. ins.
24. 37·6 cub. ins.
25. 21·6 cub. ins.
26. 9·75 cub. ins.

Miscellaneous Examples II. PAGES 65, 66.

2. 12 sq. ins., 1·2 sq. ins.
3. 5·4 sq. ins.
4. 4·1 sq. ins.
5. 1·5 sq. ins.
6. 9·9 sq. ins.
7. 152 sq. cms.
8. 3560 cub. cms.
9. 22200 sq. ft.
10. 40·7 sq. cms.
12. Area is 5·93 sq. ins., angles are 71·8°, 61·8°, 46·4°.

CHAPTER III.

Exercises. PAGES 69–114.

1. 113 ft. N.W.; 160 ft. N.; 113 ft. N.E.; 0.
2. (i) 4·95′, 0·6° W. of S.; (ii) 6·97′, 11·4° W. of S.
7. 14·3 cms., 23·2° S. of E.
9. 3·69 m./hr. S.W.; 3·7 m./hr., 25·9° S. of W.; 1·97 m./hr., 32·2° N. of W. Speeds 3·69; 4; 3·86 m./hr.
11. 5·83 ft./sec., 31° E. of N.
12. 36·9° with the up-stream line.
13. 24·3° with the up-stream line.
14. 258 m./hr., 22·8° N. of E.
15. 16·6 m./hr., 25·2° W. of S.
17. 14·3 m./hr. from 36·5° N. of E.
18. 17 m./hr. (nearly).
19. From 28·4° N. of E. and 15·6° W. of N.
20. 3·3 miles.
21. 13·5 m./hr. 47·5 E. of S.
22. 9·2 m./hr., 14·7° E. of S.; 15·7 m./hr., 14·7° E. of S.
23. The total accelerations are 17 ft./sec., 48·3° S. of E., 26·1 ft./sec. E., 16 ft./sec., 45° N. of E. and 0.
24. 41·9 ft. per sec. per sec. at 22·5° with its own direction of motion.
26. 7·07 ft. each.
27. 10·6 and 5·5 ft.
28. 27·9 and 16·1 ft. per sec. per sec.
29. 4·93 and 0·83 ft. per sec. per sec.
30. 47 miles.
31. 29·2 ft. per sec. per sec.
32. 1·61 ft. per sec. per sec. down the incline.
33. 2·31 ft. per sec. per sec. making 43·4° with the vertical.
42. 0·85″ from the centre.
43. 1·1″ and 0·8″ from the sides containing the second and third, and the third and fourth masses respectively.
48. 5·93 cms. from origin.
49. 4″ from origin.
53. 1·92″ and 0·29″.
57. 0·608 of radius from centre.

58. 0·84″ from centre of circle.
66. 0·19″ from the centre of the 3″ circle away from the hole.
69. 0·33″ from the centre of the rectangle away from the hole.
70. 0·44″ from centre of rectangle.
71. 0·57″ from centre of rectangle.

Miscellaneous Examples III. PAGES 115, 116.

3. 26·8 ft. per sec. at 52·6° with the horizontal.
4. 11·2 m./hr., 49·1° N. of W. **5.** Coordinates are 2·42 and 3·54 cms.
6. Coordinates are 2·32 and 3·88 cms. **8.** 2·25″ from centre.
9. 0·78 and 0·61 of the radius from AB and BC respectively.
11. 7·17 ft. from base of block.

CHAPTER IV.

Exercises. PAGES 119–166.

2. 166 lbs. wt.
3. (i) $\theta=\phi=36{\cdot}9^\circ$. (ii) $R=8{\cdot}7$ lbs. wt., $\phi=30^\circ$.
(iii) $\phi=34{\cdot}05^\circ$, $\theta=44{\cdot}4^\circ$. (iv) $Q=8{\cdot}18$ lbs. wt., $\phi=48^\circ$.
(v) $P=14{\cdot}2$ lbs. wt. $\phi=76{\cdot}3^\circ$. (vi) $P=5{\cdot}38$, $Q=4{\cdot}12$ lbs. wt.
(vii) $\theta=33{\cdot}8^\circ$, $Q=7{\cdot}76$ lbs. wt. (viii) $P=6{\cdot}54$.
(ix) $P=10{\cdot}73$, $Q=9$. (x) $R=5{\cdot}66$, $Q=3{\cdot}35$.
4. 95·5°, 134·7°, 129·8°. **5.** 59 and 107 grms. wt. respectively.
6. (i) 12·7 and 20·8 lbs. wt. respectively.
7. 2·98 cwts. at A and 1·57 at B.

8.

A	5,340	5,880	8,980	12,600	14,800	30,600
B	8,790	8,720	7,560	9,080	11,400	29,000

12. 111 making 112·75° with the 30 force.
13. $R=37{\cdot}4$, $S=6{\cdot}3$. **14.** 0·2 and 1·67 lbs. wt. respectively.
15. 7·54 and 12·52 lbs. wt. respectively.
17. AB makes 21·3° with the vertical, the pull on A is 10·25 lbs. wt.
19. $W=32{\cdot}8$ lbs. wt. **20.** 11 lbs. wt. at 35·3° with the vertical.
21. 1·29 cwts. **22.** 20·5 and 36 lbs. wt.
23. Through one end of a draw a line parallel to OB, describe a circle of radius c with the other end of a as centre. The circle cuts the line drawn parallel to OB in two points, hence c has one of two directions. If c has a direction perpendicular to OB the magnitude of c will be the least possible.
24. (i) 13·4 and 44·8 lbs. wt. (ii) Diminished by 5·3 and 7·3 lbs. wt.
(iii) Increased by 5·3 and 7·3 lbs. wt. (iv) 18·3 lbs. wt.

25. 20·7 kilogrms. wt., 40·1° E. of N.

26. 6·5 lbs. wt. at 67·2° with the 23 force.

30. 18·2 lbs. wt. towards a point 33·7° N. of E.

31. 23·8 lbs. wt. at 79·1° with the 10 lb. wt. force.

32. 20·7 lbs. wt. bisecting the angle *ABC*.

33. 12·7 lbs. wt. at 45·05° with *AB*. **34.** 9·33 and 9·66 kilogrms. wt.

35. 1·29 and 2·5 kilogrms. wt. **36.** 1·77 and 3·42 kilogrms. wt.

38. 0·4 cwts.

39. First draw the plane. Then set off *AC* vertically downwards for the weight of the body and draw *AB* and *BC* parallel and perpendicular to the plane. Set off *AE* along *AB* for the pull parallel to the plane, and then draw *ED* at 15° with the horizontal cutting *CB* at *D*. Scale *DE* and *DC*; they give 10·9 and 14·3 cwts. as the required forces.

40. Draw parallel to γ a radius of the circle and measure the length of the arc to the highest point (1·22′).

41. The slope is 0·41 the angle 21·8°.

42. 24·3°. **43.** 39°; 16·1 lbs. wt. **44.** 3·39 cwts.

45. 4·5 and 3 kilogrms. wt.

46. 15·6 lbs. wt. at 45° with the vertical.

47. 9·3 and 19·9 lbs. wt. bisecting the angles.

48. (i) 388,1300 and 388 grms. wt. bisecting the angles.
(ii) 1449,1299 and 1449 grms. wt. bisecting the angles.

49. 5·71 tons wt. in *AC*, 4·28 in *BC*. **50.** 8·61 tons wt. in *BC*.

51. 2·33 tons. wt. in *BC*, 4·46 in *AC*.

53. 6·39 tons wt. in *AC*, 3·83 in *BC*.

54. 4·3 tons wt. in *AC*, 0·92 in *BC*.

55. 1·89 kilogrms. wt. in *AC*, 1·19 in *AB*, 0·98 in *BC*, 0·68 kilogrms. wt. reaction at *B*, 1·62 at *A*.

56. 4·44 tons wt. in *AB*, 9·62 in *AC*.

57. 2·8 tons wt. in *AB*, 15·9 in *AC*.

59. Reaction of nail = weight of picture; tension = 4·62 lbs. wt.

61. 1·72 cwts. in *BC*, 3·5 in *AC*, *C* being below *AB*.

62. Stress in *BA*, 5·29 tons wt. **63.** Tension is 15·5 lbs. wt.

64. First find the tension in the chain by drawing from the ends of the vertical load vector a horizontal line and one making 30° with the vertical. Find the total pull at *B* of the two parts of the chain and then the forces in *AB* and *CB* which are in equilibrium with this. Stress in *AB* is 6·6 cwts., in *CB* 17·3 cwts.

65. 18·18, 42·1° N. of E. **67.** 10·02 and 9·92.

69. 129, 321, 453, 483 lbs. wt. respectively.

70. 11 and 5·69 lbs. wt. respectively.

72. (i) 421 lbs. wt. along the rail, side thrust 201.
(ii) 421 lbs. wt. along the rails, side thrust 21.

73. (i) 5·3 and 10·55 lbs. wt. (ii) 10·55 and 5·3 lbs. wt. (iii) 3·7 lbs. at each.

75. $P=3\cdot67$ lbs. wt., reaction = 11 lbs. wt. Component parallel to plane = 3·67 lbs. wt., vertical component of reaction = 10·4 lbs. wt.

76. 27 lbs. wt., 75·3 lbs. wt. **77.** 132·7 lbs. wt., 55·8 lbs. wt.

78. The wedge is in equilibrium under the horizontal push of 28 lbs. wt., the reaction of the block and a vertical force equal to the table reaction less the weight of the wedge. Draw a horizontal line representing to scale the push of 28 lbs. wt. From its end points draw lines vertical and perpendicular to face of wedge supporting the block, the latter gives 59·6 lbs. wt. as the reaction of the block on the wedge. The former gives 52·5 as the vertical force on the wedge, and this consists of 18 lbs. downwards and the reaction of the table upwards. The table reaction is therefore 70·5 upwards.

79. 12·2 and 43·5 lbs. wt. respectively.

84. 199 lbs. wt. **87.** 5·85, 12·5, 37·5 lbs. wt. **89.** 2·35 lbs. wt.

90. 2·36 lbs. wt. at 43·7° with the vertical.

91. 0·252 cwts. at wall, 0·564 at ground, making 63·45° with the horizontal.

92. 0·217 cwts. at wall, reaction at ground makes 23·25° with the vertical.

93. Tension is 22·1 kilogrms. wt.

97. For the 60° plane the reaction is 29·9 lbs. wt., the reaction of the hinge is 60·8 lbs. wt., making 25·2° with the vertical.

98. 1·25 cwts. at the top and 1·66 cwts. at the bottom, the latter making 51·6° with the horizontal.

99. 521 lbs. wt. at hinge, making 24° with the beam, and 265 lbs. wt. at the top.

100. $Q=212$ lbs. wt., reaction at C 232 lbs. wt. passing through the intersection of P and Q.

101. 42 lbs. wt. at plane and 48 at the peg, making 25·6° with the vertical.

102. 83·1° with the vertical. **103.** 50·8° with the vertical.

104. 42·7° with the vertical.

105. 5·1° with the horizontal; reactions of the planes are 3·66 and 2·59 cwts.

CHAPTER V.

Exercises. Pages 172–206.

2. Resultant of magnitude 3·62 lbs. wt. inclined at 13·8° with BC, the axis cuts AB produced at 1″ from B.

3. Resultant of magnitude 8·99 lbs. wt., making 48·7° with Ox and cutting it at $-0\cdot31$ units from origin.

8. Magnitude 6·62 lbs. wt. making 78·3° with the first spoke, axis cuts first spoke at 4·6″ from its point of contact with the hub.

11. 12 lbs. wt. at 6·42″ from the smaller weight.

12. 3·86″ from the left end weight.

13. First find the resultant of the first two forces and the given equilibrant. The third rope is 3·9″ from the mid-point and nearer the smaller weight.

14. Draw the link polygon for the weights W_1 and W_2, mark the points where the first link cuts the known axis of the resultant, and where the last link cuts the axis of W_3. A line through the pole of the vector polygon parallel to the line joining these two points determines the magnitude of W_3 (19·75 lbs. wt.).

15. – 5 lbs. wt. at 9 ft. from first force and 6 ft. from second.

16. – 22 lbs. wt. at 17·1″ from the end.

17. Magnitude 27·2 lbs. wt., the axis making 22·1° with *PQ* and cutting it at 0·47″ from *P*.

18. 1 cwt. at 0·72 yds. from first and 0·28 yds. from second force.

19. 197″ from leading wheel. **25.** 4·73 and 5·97 tons.

26. 23·65 and 22·35 cwts. **27.** 39·4 and 6·6 cwts.

28. No; reaction 1·3 cwts. downwards.

31. 23·7 cwts. at roller, reaction at pin makes 41·6° with beam and is of magnitude 31·5 cwts.

33. 185·7 lbs. wt. at plate, 273 lbs. wt. at hinge, making 42·9° with vertical.

34. 364 lbs. wt. at plate, 472 lbs. wt. at hinge, making 50·5° with vertical.

35. (i) 327 lbs. wt. at plate, 499 lbs. wt. at hinge, making 54·5° with the gate post.

37. Reaction at cylinder 1·82 cwts., reaction at hinge 1·22 cwts., making 51·9° with the horizontal.

38. 348 lbs. wt., making 15·2° with the vertical.

46. (i) Perpendicular to the pole with a force 1·81 times the weight of the pole; (ii) a force of 1·86 times the weight of the pole at an angle of 76·9° with the pole.

53. 10·3 tons wt., making 16·1° with the vertical.

CHAPTER VI.

Exercises. PAGES 210–251.

2. *RQ* and *PR* in tension; stress 9·82 lbs. wt. in each. *PQ* in compression; stress 4·91 lbs. wt.

3. Reaction at *P* is 12·45 lbs. wt., stress in $PQ = 3{\cdot}72$ lbs. wt. and at $QR = 5{\cdot}08$ lbs. wt.

5. A tensile stress of 4·97 lbs. wt. in the lower bars; a compressive stress in the horizontal bar of 14 lbs. wt.

6. Lower bars have a tensile stress of 4·97 lbs. wt., the vertical bar one of 14 lbs. wt.

7. The diagonal bar makes 4·75° with the horizontal; the stresses in the two lower bars are 4·43 and 5·42 lbs. wt.

9. The stresses in *AB*, *BC* and *CA* are 21·7, 56·3 and 27·3 lbs. wt.

11. Tensile stress in *PQ* of 2 tons wt., compressive stress in *PT* of 1·73 tons wt., compressive stress in *TS* of 1·15 tons wt.

12. Compressive stress in PQ of 2·31 tons wt., in QR a tensile stress of 1·15 tons wt.

13. Stress in PQ is 2·31 tons wt. **14.** Stress in PU is 8·66 tons wt.

23. 33·1 lbs. wt. along the line joining P to the mid-point of QR.

30. Average compressive stress in AB is 3·93 lbs. wt.

CHAPTER VII.

Exercises. PAGES 256–280.

1. $\mu=\frac{1}{2}$, $\epsilon=26\cdot6^\circ$.

2. Reaction 27·4 ozs., making 18·4° with the vertical.

3. (i) Yes ; (ii) no. **4.** $\mu=0\cdot447$, $\epsilon=24\cdot1^\circ$.

5. 467 grms. wt. **8.** 4·09 and 3·79 lbs. wt.

9. Least force 11·7 lbs. wt. at 32·4° with the horizontal, corresponding friction is 0·98 lbs. wt.

10. Yes ; 15·5° with the vertical.

11. 5·64 cwts. at 19·5° with the vertical, horizontal resistance 1·88 cwts.

12. Least pull is 12·8 lbs. wt. at 20·3° with the horizontal.

13. Angle of friction 36·9°, reaction 8 lbs. wt.

14. (*a*) 3·5. (*b*) 4·14 cwts. (*c*) 2·98. (*d*) At 25° with the horizontal.

15. 6·23 kilogrms. wt. is the least force, 17·81 in the opposite sense to the 5 force.

16. 0·78 tons wt., making 38·7° with the vertical; (*a*) 1·02 tons wt.; (*b*) 0·83 tons wt.

17. Greatest force 1·31 kilogrms. wt.

18. 3·32 cwts. **19.** 6·43, 3·46 cwts.

20. 11, 2·85 and 7·78 lbs. wt. For the friction in the three cases resolve the total reaction of the surface into two components along and perpendicular to the plane, the former components give the friction.

21. 10·5 and 14·7 lbs. wt. **22.** 48·5 lbs. wt.

23. 10·3°. **24.** 2·85 ft., least tension 0·896 cwts.

25. 43·6°. **27.** $\mu=0\cdot14$.

28. The M.C. divides ladder in ratio 13/100.

29. 14·29 lbs. wt. ; $\mu=0\cdot866$.

31. 8·8° with the horizontal.

32. The M.C. divides the beam in ratio 342/100.

33. The M.C. divides the beam in ratio 211/100.

34. The M.C. divides the beam in ratio 323/100.

36. The M.C. is nearly at the mid-point.

38. 63° with the vertical.

39. 76·7° with the vertical.

41. 70° with the vertical, no other possible position for limiting equilibrium.

CHAPTER VIII.

Exercises. PAGES 284–302.

3. 475 lbs. ft.
6. (i) – 38·7. (ii) 13·7. (iii) – 8·5 tons ft.
8. 5450 tons inches. **9.** 13700 tons inches.
11. Take the pole 10 cms. from the force vector.
12. Measure the line giving the momental area on the kilogramme weight scale.
14. 15·7 lbs. inches. **16.** – 11·4 lbs. inches.
20. 2·67 cms. from *CD*, 4·2 from *AD*.
21. 6·66 cms. from *CD*, 2·05 from *AD* outside the square.
29. 5·4 cms. from lower centre. **30.** 5·62 cms. from lower centre.

CHAPTER IX.

Exercises. PAGES 315–332.

1. B.M. is 4000 lbs. ft., S.F. is 500 lbs. wt.
2. B.M's are 22,500, 10,000 and 5000 lbs. ft.
S.F's are 2000 lbs. wt. at 8 ft.
1700 lbs. wt. at just under 15 ft.
1000 lbs. wt. at just over 15 ft.
1000 lbs. wt. at 20 ft.
3. B.M. at 8 ft. is 21·2 tons ft.
S.F. at 8 ft is 1·9 tons wt.
4. At 7 ft. from *Q* the B.M. is 21,100 lbs. ft. and the S.F. is 1500 just under, and 750 just over 7 ft. from *Q*.

CHAPTER XI.

Exercises. PAGES 354–367.

1. 1420 ft.-lbs. **2.** 16,000 ft.-lbs. **3.** 18,500 inch-lbs.
4. 1930 ft.-lbs. **5.** 48·4 ft.-cwts. **6.** 117 ft.-cwts.
8. 799 ft.-lbs. (independent of the radius). **9.** 3470 ft.-lbs.
10. 193 ft.-lbs. against gravity, 57 against friction, 250 by the equilibrating force. Total work is zero.
11. 278 and 367 ft.-lbs. **12.** 765 ft.-lbs.
13. 335 ft.-lbs. **14.** 2,520,000 ft.-lbs.
15. 20·7 ft.-lbs. round *B* and 66·2 round *C*.
16. 224,500 ft.-lbs. **18.** 303 ft.-lbs. **19.** 54·6 inch-lbs.
20. 57·9 inch-lbs. **21.** 19·3 ft.-lbs.

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